Towards empirical assessments of controlled cointegrated models

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Abstract

This paper approaches stabilisation policy and control theory in the cointegrated Vector AutoRegressive (VAR) model from the perspective of an applied econometrician. We show that the observables generated by the policy can be seen as driven by a Vector AutoRegressive Moving-Average (VARMA) model, which can be given a Structural VAR interpretation. This allows the econometrician to identify and assess the policy. Exploring further the mechanisms involved in policy implementation, we introduce a data-driven approach for classifying intermediate and final policy targets within a model framework. The practicality and effectiveness of this procedure are demonstrated through an analysis of New Zealand's monetary policy data.

JEL classification codes: C32, C54, E52

Keywords: Cointegrated vector autoregressive model; Structural Vector Autoregressions; Control policy; Counterfactual analysis; Policy targets; Monetary policy.

1 Introduction

This paper explores control theory within the framework of a cointegrated vector autoregressive (CVAR) model, viewed from the standpoint of an econometrician aiming to conduct counterfactual analysis or to assess ex post the presence and effectiveness of stabilisation policies. This paper builds extensively on Johansen and Juselius (2001), hereafter referred to as JJ.

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JJ pioneered a method for policy simulation analysis using a CVAR model, and numerous empirical studies have applied this method to macroeconomic and financial time series data; see Christensen and Nielsen (2009), Carlucci and Montaruli (2014), Boug *et al.* (2024), and Castle and Kurita (2024), among others. In addition, Chevillon and Kurita (2024) applied the method to climate data and conducted various counterfactual policy simulation analyses, demonstrating the usefulness of CVAR-based policy analysis in the context of climate econometrics. The articles by Johansen and Juselius in this issue present a revised and extended version of their previous work, JJ, expounding the core concepts of their method.

This papers aim to take the stance of the applied econometrician willing to assess policies within the class considered by JJ. We can of course consider the counterfactual *retroactive* or *prospective* implementation of a policy, which the articles mentioned above have done. But here, we are asking whether the econometrician can estimate and assess the policy, after it has been implemented. This implies considering what the observables are, and how to estimate the parameters. Doing so, we find that the JJ setting as presented in their original paper is not easy to reconcile with empirical analysis. In particular, we find issues determining which is the observed data, and hence estimating the data generating process. Yet, with the help of Rambachan and Shephard (2021), we identify a different interpretation of JJ's mathematical results. Here, we show the data, post policy implementation, can be seen as generated by a vector autoregressive moving-average process, or a VARMA$(p, 1)$ process, which can be given an structural vector equilibrium correction (SVEC) interpretation where the lagged MA(1) innovation constitutes the "policy" shock. This allows us to conclude that the applied econometrician can ex post identify the policy.

This allows to go further, and explore more at depth the mechanisms presiding over the control policy, with the intention of delineating how the authorities can put their policy in place. For this we focus on the situation where they cannot directly control the target through their instrument, but must rely on market forces through an intermediate target. This is a situation studied by JJ. While their definitions of intermediate and final targets are clear and unproblematic in theoretical contexts, challenges arise when these concepts are applied empirically. Specifically, distinguishing between intermediate and final policy targets among a set of candidate variables is complex in practice, particularly when analysing the time series data of candidate variables in a multivariate CVAR system. Although insights from economic theory are valuable, they are often insufficient to justify an a priori distinction in most cases. One of the aims of this paper is to propose a procedure for identifying intermediate and final policy targets in empirical contexts. An empirical illustration of the proposed procedure is provided by modeling the macroeconomic time series data of New Zealand.

New Zealand was chosen for the following reasons: (i) New Zealand is a front-runner in inflation targeting policy, being the first country to formally adopt the policy in the early

1990s and is well-known for its successful implementation over the past three decades; and (*ii*) the Reserve Bank of New Zealand (RBNZ) has published long-term time series data on inflation expectations. While the suggested procedure can be applied to time series from other economies, New Zealand's data are particularly suitable for demonstrating the usefulness of our procedure in the context of policy simulation analysis. For a preceding empirical illustration of applying a cointegrated method to New Zealand's time series data, see Choo and Kurita (2016), *inter alia*.

The rest of this paper is organised into four sections. Section 2 briefly reviews control theory within a CVAR system and then considers issues faced by econometricians working with the data generated after a policy has been implemented. Section 3 addresses the issue of identifying intermediate and final policy targets and considers an empirical procedure. Section 4 provides an empirical illustration of the procedure. Finally, Section 5 presents concluding remarks. All econometric analyses in this paper were conducted using *Cats* (Doornik and Juselius, 2023), *Ox* (Doornik, 2023) and *PcGive* (Doornik and Hendry, 2023).

2 Inference in controlled cointegrated systems

This section revisits control theory within the context of a CVAR model through the perspective of the econometrician willing to estimate a model and perform counterfactual analysis. We begin by reviewing JJ's control theory and then discuss various methodological issues related to counterfactual policy analysis using a CVAR system.

2.1 CVAR-based control theory

We start by providing a brief review of control theory in a CVAR model for I(1) nonstationary time series data; for further details of the model, refer to Johansen (1988, 1996), Juselius (2006) and Hunter *et al.* (2017). Let X_t be a p-dimensional vector of time series which is represented as the following trend-restricted CVAR(k) model conditional on $X_{-k+1}, \ldots, X_0$:

$$\Delta X_t = \alpha \left(\beta', \rho\right) \begin{pmatrix} X_{t-1} \\ t \end{pmatrix} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \tau + \varepsilon_t, \quad \text{for} \quad t = 1, ..., T, \tag{1}$$

where ε_t is a martingale-difference sequence with a positive definite variance matrix $\Omega \in \mathbf{R}^{p\times p}$, a process satisfying a class of assumptions provided by Kurita and Nielsen (2019). The parameters of (1) are defined as $\alpha, \beta \in \mathbf{R}^{p\times r}$ for $r < p$, $\Gamma_i \in \mathbf{R}^{p\times p}$, $\rho \in \mathbf{R}^r$ and $\tau \in \mathbf{R}^p$. The parameters α (adjustment or loading vectors) and β (cointegrating vectors) are assumed to be of full rank r (the cointegration rank). Let their orthogonal complements $\alpha_\perp, \beta_\perp \in \mathbf{R}^{p\times(p-r)}$ of full rank $p - r$, so that the equality $\alpha'_\perp \alpha = \beta'_\perp \beta = 0$ holds along with the non-singularity of the two matrices $(\alpha, \alpha_\perp)$ and $(\beta, \beta_\perp)$. In order to justify I(1) CVAR analysis rather than I(2) or higher order degrees of integration, we assume that $\alpha'_\perp \Gamma \beta_\perp$ is of full rank $p - r$ for $\Gamma = I_p - \sum_{i=1}^{k-1} \Gamma_i$. In the development of control theory in the next subsection, it is also essential to introduce here $C = \beta_\perp(\alpha'_\perp \Gamma \beta_\perp)^{-1}\alpha'_\perp$, known as the impact matrix in the Granger-Johansen representation.

The control theory developed by JJ considers a policy aiming at stabilising a subset of the system variables, or a linear combination thereof, so they become stationary around a specified mean. We simplify here the model (1) into a constant-restricted model with $k = 1$ to make the required argument straightforward (but prove our results in the general case in the Appendix):

$$\Delta X_t = \alpha(\beta' X_{t-1} - \mu) + \varepsilon_t, \quad \text{for} \quad t = 1, ..., T, \tag{2}$$

for $\mu \in \mathbf{R}^r$. Equation (2) provides a basis for a review of the theory.

We now introduce two policy matrices $a, b \in \mathbf{R}^{p \times m}$ for $m + r < p$. The matrix a is associated with the selection of policy instruments, $a'X_t$, while the matrix b pertains to the selection of policy targets, $b'X_t$. The aim of economic policy is to stabilise $b'X_t$ using $a'X_t$, which means making $b'X_t$ stationary with mean b^*, the policy target level, through the use of $a'X_t$. Achieving this requires a t-timed contemporaneous policy intervention, represented by $\kappa' X_t - \kappa^*$ for $\kappa \in \mathbf{R}^{p \times m}$ and $\kappa^* \in \mathbf{R}^m$. Policy implementation replaces X_t with X_t^{ctr}, dubbed the 'controlled process', which is defined as

$$X_t^{ctr} = X_t + \overline{a}\left(\kappa' X_t - \kappa^*\right), \tag{3}$$

for $\overline{a} = a(a'a)^{-1}$.

Given X_t^{ctr}, assuming market dynamics are not modified by the intervention, equation (2) generates a *new* series X_t^{new}, with the requirement that $b'X_t^{new}$ is stationary with mean b^*. The overall process is hence two-staged, so that from a policy inception at date t_0, and letting $\nu : x \to \overline{a}\left(\kappa' x - \kappa^*\right)$,

$$X_{t_0} \to \underbrace{X_{t_0}^{ctr} = X_{t_0} + \nu\left(X_{t_0}\right)}_{\text{(Policy)}} \to \underbrace{X_{t_0+1}^{new} = \left(I_p + \alpha\beta'\right) X_{t_0}^{ctr} - \alpha'\mu + \varepsilon_{t_0+1}}_{\text{(Ecosystem)}}$$

$$\to \underbrace{X_{t_0+1}^{ctr} = X_{t_0+1}^{new} + \nu\left(X_{t_0+1}\right)}_{\text{(Policy)}} \to ....$$

To determine the parameters of the policy rule that achieve stabilisation, we see that equation (2) implies that the long run response of the system satisfies

$$X_\infty \equiv \lim_{h \to \infty} \mathbf{E}(X_{t_0+h} \,| X_{t_0}) = CX_{t_0} + \alpha(\beta'\alpha)^{-1}\mu.$$

The response of the economic process to the policy introduction is therefore, in the directions defined by the policy target b, $\lim_{h\to\infty} \mathbf{E}\left(b' X_{t_0+h}^{new} \,\middle|\, X_{t_0}^{ctr}\right) = b' \left\{ C \left[X_{t_0}^{ctr}\right] + \alpha(\beta'\alpha)^{-1}\mu \right\}$. Hence

$b'X_t^{new}$ having been stationarised by the intervention around b^* means that, in terms of the original variable,

$$b^* = b' \left\{ C \left[X_t + \overline{a} \left(\kappa' X_t - \kappa^* \right) \right] + \alpha(\beta'\alpha)^{-1}\mu \right\}.$$

Hence, if $det(b'Ca) \neq 0$ is satisfied, the policy rule satisfies

$$\kappa' X_t - \kappa^* = -(b'C\overline{a})^{-1} \left[b'CX_t - b^* + b'\alpha(\beta'\alpha)^{-1}\mu) \right],$$

so that $\kappa' = -(b'C\overline{a})^{-1}b'C$ and $\kappa^* = -(b'C\overline{a})^{-1} \left[b^* - b'\alpha(\beta'\alpha)^{-1}\mu) \right]$ are the solutions. We thus refer to $det(b'Ca) \neq 0$ as the controllability condition hereafter.

The identity $C = I_p - \alpha \left(\beta'\alpha \right)^{-1} \beta'$ then leads to the following important equation

$$\kappa' X_t - \kappa^* = (b'C\overline{a})^{-1} \left[b'\alpha(\beta'\alpha)^{-1}(\beta' X_t - \mu) - (b'X_t - b^*) \right], \tag{4}$$

i.e., the policy rule constitutes of weighted average of two forms of disequilibria, given respectively by $\beta' X_t - \mu$, a vector of deviations from the long-run relationships, and $b'X_t - b^*$, the discrepancy between the actual and desired targets.

As the policy needs to be implemented every period, $X_t^{ctr} = X_t^{new} + \overline{a} \left(\kappa' X_t^{new} - \kappa^* \right)$ for all $t > t_0$ and JJ derive the corresponding *new* dynamics:

$$\Delta X_{t+1}^{new} = \left[\alpha, \left(I_p + \alpha\beta' \right) \overline{a} \right] \begin{bmatrix} \beta' X_t^{new} - \mu \\ \kappa' X_t^{new} - \kappa_0 \end{bmatrix} + \varepsilon_{t+1}. \tag{5}$$

The *new* system is characterised by an additional cointegration relation corresponding to the implemented policy.

2.2 The Econometrician's problem

We now consider JJ's analysis from the perspective of the econometrician who aims to identify the policy and derive a counterfactual analysis. We analyse in turn the elements this econometrician must consider.

2.2.1 Timing and Observables

The principle of JJ's approach to policy is that the control rule is applied at each point in time, thus generating two new processes (X_t^{new}, X_t^{ctr}) which accord to a specific timing. Let us stress it again as it is important for our discussion.

1. At the beginning of period t, the authority observes the process that is generated by market forces. We denote it by X_t^{new} for simplicity (at $t = t_0$, we let $X_{t_0}^{new} = X_{t_0}$). The authority then chooses an intervention $\nu\left(\cdot\right)$ that modifies X_t^{new} and generates the controlled process.

$$X_t^{ctr} = X_t^{new} + \nu\left(X_t^{new} \right). \tag{6}$$

2. The market at time $t+1$ generates the next value X_{t+1}^{new} of the process according to (6),
$$X_{t+1}^{new} = (I_p + \alpha\beta') X_t^{ctr} - \alpha'\mu + \varepsilon_{t+1}. \tag{7}$$
3. The authority intervenes again and sets $X_{t+1}^{ctr} = X_{t+1}^{new} + \overline{a}\left(\kappa' X_{t+1}^{new} - \kappa_0\right)$, and so on.

In JJ's framework, the intervention is contemporaneous so X_t^{new} is immediately transformed into X_t^{ctr}, and this is the process that market forces then work with to generate the next period's observations. Given that the policy is implemented every period (between inception and termination), the process observed by the econometrician must be X_{t+1}^{ctr}. The 'new' process X_{t+1}^{new} corresponds to a latent state, as it is immediately transformed by the authority and never actually holds.

An alternative timing for decisions is possible, which may render both processes observables. This would require introducing subperiods at t, so for instance if we think of Federal fund rates as policy instruments, then markets and the econometrician observe X_t^{new} at the beginning of the period (a month or a quarter), the decision/intervention is then made at the FOMC meeting during the period, this is when X_t^{ctr} is generated and becomes the new value for the whole vector of t-timed observables. Next period's outcome becomes available in its first half, X_{t+1}^{new} then X_{t+1}^{ctr} later in the period. An issue with this interpretation is that X_t^{new} and X_t^{ctr} correspond to the same set of variables, so that if the Federal Funds rate is part of X_t^{new}, the observed value is contingent on the policy that is being followed, it cannot be set both by markets and authorities at once, or we are really dealing with two distinct concepts that must be represented by two different variables. The only solution would be to assume that the policy instrument is only set by the authority, so Ω, the variance-covariance of ε_t is singular, with zero variance in the direction of a, i.e., in model (1):
$$Var\left(a'\varepsilon_t\right) = 0.$$
In the example of the Federal Funds rate, the variance of their innovations must then be zero.

This alternative timing does not correspond to the system's assumptions, absent the policy, so it cannot hold. It follows that the process that is observable by the econometrician should *a priori* be X_t^{ctr}, and that X_t^{new} should be latent.

2.2.2 What's wrong with the controlled process?

Let us entertain the consequences of the discussion above, where the policy generates a unique set of observables, X_t^{ctr}. While the theory has established that X_t^{new} exhibits an increased rank of cointegration, as in expression (5), and that $b'X_t^{new}$ is stationary about b^*, the dynamics of X_t^{ctr} is atypical. Indeed, the natural control rule derived by JJ is such that

$\kappa'\alpha = 0$, and, $I_m + \kappa'\overline{a} = 0$. Notice in this case that, contrary to the latent X_t^{new}, whose rank of cointegration is $r + m$, the observable X_t^{ctr} maintains a cointegration rank equal to r, with modified reduced-form errors:

$$\Delta X_{t+1}^{ctr} = \alpha \left(\beta' X_t^{ctr} - \mu\right) + \left(I_p + \overline{a}\kappa'\right) \varepsilon_{t+1}, \tag{8}$$

which is compatible with

$$\kappa' x_{t+1}^{ctr} = \kappa_0. \tag{9}$$

Thus, in terms of the controlled variable X_t^{ctr}, the cointegration properties are not altered compared to the process without control. Yet, the innovations to X_{t+1}^{ctr} in (8) also exhibit a singular variance-covariance as

$$\det\left[\left(I_p + \overline{a}\kappa'\right) \Omega \left(I_p + \overline{a}\kappa'\right)'\right] = 0, \tag{10}$$

see the proof in the Appendix. This reduced rank of the variance-covariance of the innovations implies that the controlled process exhibits peculiar dynamics. This is clear from the control policy which ensures that at all times

$$b' \left[C X_t^{ctr} + \alpha \left(\beta'\alpha\right)^{-1} \mu\right] = b^*,$$

i.e. a linear combination of X_t^{ctr} remains constant at every period. Rewriting the above, we see that

$$b' X_t^{ctr} - b^* = b'\alpha \left(\beta'\alpha\right)^{-1} \left(\beta' X_t^{ctr} - \mu\right).$$

Hence $\beta' X_t^{ctr} - \mu \sim \mathsf{I}(0)$ implies that $b' X_t^{ctr} - b^* \sim \mathsf{I}(0)$ and the two are collinear. The increased rank of cointegration in X_t^{new} is only implicitly present in X_t^{ctr} since the second cointegration relation is directly proportional to the first, for this process. The reduced rank of the covariance of the innovation for X_t^{ctr} implies that, when this is the observable process, cointegration analysis will lead misleading results for the econometrician as we show next by simulation.

2.3 Monte Carlo evidence

We now document the extent to which degenerate dynamics for X_t^{ctr} impair inference on the cointegrating rank, and hence on the policy evidence. For this we consider the situation where an authority implements a univariate policy ($m = 1$). To presence some plausibility of policy control, we assume the parameters of the CVAR are estimated (all ranks and dimensions are known, including the directions of α and β) when forming the policy. We then consider the evidence the econometricians obtain on the rank of cointegration for the original, controlled and new processes. Throughout we let the data generation process (DGP) follow a CVAR(1) but modify system dimensions and cointegration ranks. See the Appendix for further details

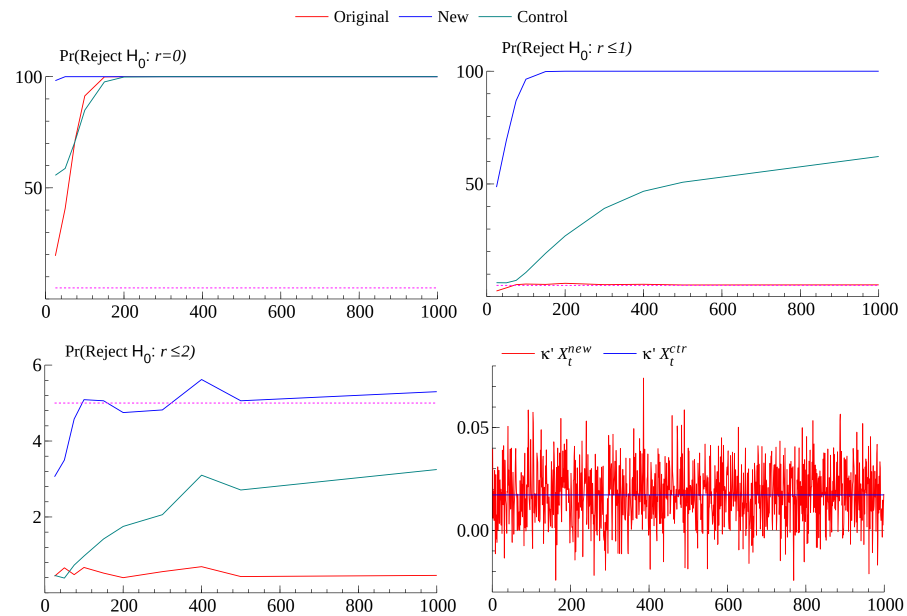

Figure 1: Panels (a)-(c): Rejection frequencies, at the 5% nominal level, of the Trace test of a cointegration rank r in a 3-variate system as a function of the sample size (horizontal axis). Each panel corresponds to a different hypothesized value of r when the truth if 1 for the original data, and 2 for the *new* data. Distributions are obtained by simulation over 10,000 replications. Panel (d) on the bottom right presents one realization of the processes over a sample of dimension 1000.

of DGPs employed in the Monte Carlo simulation. Using 10,000 Monte Carlo replications we first report the rejection frequencies of the null of a given rank r of cointegration, using the usual cointegration trace statistics at a nominal size of 5%.

Figure 1 records such rejection frequencies for a range of null hypotheses about a three-variate system. In the data generating process, the rank of cointegration is $r = 1$ for X_t (denoted Original, in the figure), and $r = 2$ for X_t^{new} (New, in the figure). Over moderate samples of sizes greater than 200 observations, rejection frequencies are close to the nominal values under the null ($r \leq 1$ for X_t and $r \leq 2$ for X_t^{new}). The power is also high over the same sample size. By contrast, we observe massive distortions about X_t^{ctr} (Control, in the figure). Because of innovation variance rank degeneracy, the test statistic rejects on average 50% of the time for the null of 1 cointegrating relation, and 2 to 3% for the null $r \leq 2$. Treating the "controlled process" as the "new" to test for the presence of an increased rank of cointegration would therefore be misleading, with low power at $r \leq 1$ and conservative size at $r \leq 2$. Testing the correct null that $r \leq 1$ would lead to massive overrejection, owing to the singular innovation covariance matrix. To explore the issues further, we consider in Figure 2 the situation of an initial rank of cointegration $r = 2$ so the policy renders all

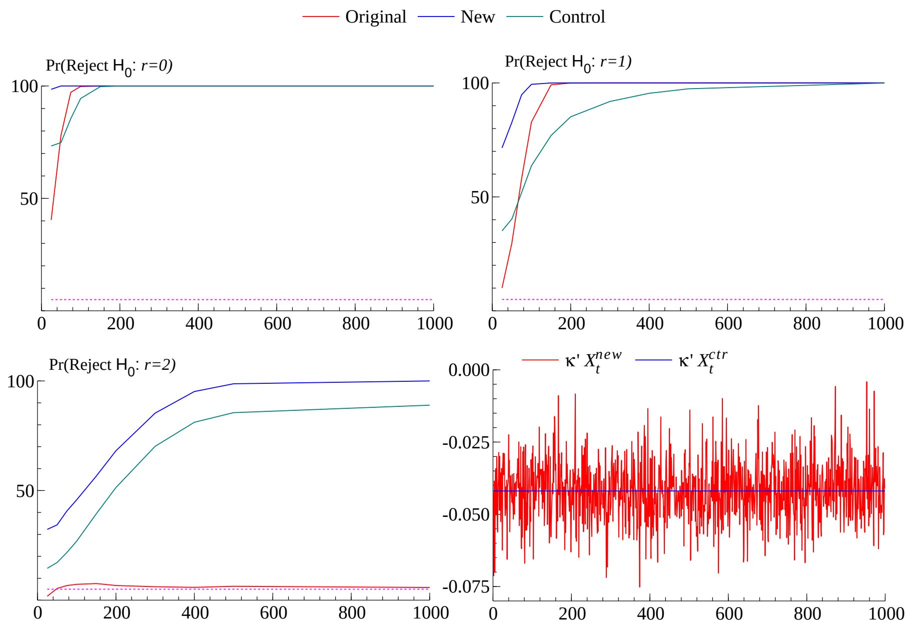

Figure 2: Panels (a)-(c): Rejection frequencies, at the 5% nominal level, of the Trace test of a cointegration rank r in a 3-variate system as a function of the sample size (horizontal axis). Each panel corresponds to a different hypothesized value of r when the truth if 2 for the original data, and 3 for the *new* data. Distributions are obtained by simulation over 10,000 replications. Panel (d) on the bottom right presents one realization of the processes over a sample of dimension 1000.

processes stationary. In this situation, inference on the controlled process is more similar to that of the new process, except that the probability not to reject $r = 2$ is higher by about 15%. In both figures, the bottom right panel presents $\kappa' X_t^{new}$ and $\kappa' X_t^{ctr}$ (for typographic reason κ is denoted $\varkappa$) and we see that the former is stationary and the latter constant.

Given that inference on the rank of cointegration follows a sequential testing procedure, we complement the previous results by a simulation where we record the frequency with which a specific rank of cointegration r_0 is selected, such that the procedure rejects $r = 0, ..., r_0 - 1$ and does not reject r_0 at the 5% nominal size (if $r_0 < p$ or this latter null is not tested). These results are presented in Figure 3 for the two DGPs considered previously, with $r = 1$ and 2. The figure shows, on the left hand side where $(p, r) = (3, 1)$, that for the original and new processes, the selection procedure achieves rates close to 95% of the correct cointegration rank. Yet for the Controlled process, this rate is about 50% for both $r_0 = 1$ or 2. On the right-hand side panels, where $(p, r) = (3, 2)$ we see that the procedures works well for the original process and for the new process. The controlled version selects $r = 3$ with a higher frequency than that of selecting $r = 2$ on the left column, but still less so than the new process.

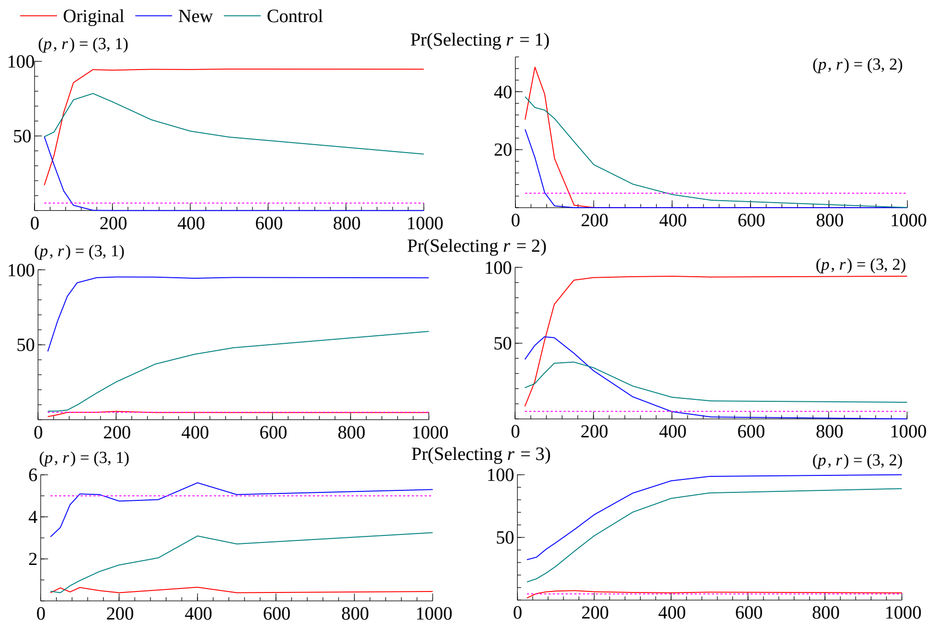

Figure 3: Probability to select a specific rank of cointegration using the sequential testing procedure based on the Trace test at the 5% asymptotic nominal size. Each row corresponds to the selection of a different rank r. The cointegration rank of the original data is 1 in the left colum, 2 in the right column. Horizontal axes record the sample size.

In order to shed more light on the reasons for the results above, Figure 4 records the distribution of the estimators of test statistics over a sample of $T = 1,000$ observations, together with that under the limiting distribution under the null. We used 10,000 Monte Carlo replications to simulate distributions. We report only the situation $(p, r) = (3, 1)$ but similar results hold for other values. We see that the difference between inference on the *new* and *controlled* processes lies essentially in that, while the Trace statistic for the null $r \leq 1$ rejects strongly for the new process, it is, for its controlled counterpart, correctly centered on the limiting distribution but with very large variability caused by the innovation variance singularity. This explains the high rejection rate we established before.

We present in the Appendix similar results for different parameter settings but with the same number of observations and replications. All of these results indicate that inference based on the controlled process is unreliable. This may seem to pose a problem for econometricians who wish to perform inference in the policy setting considered above.

2.4 A new understanding

Fortunately, while the previous analysis may imply that policy in the JJ framework will be difficult to assess empirically, we believe that it can be put to the data. Indeed, we find that

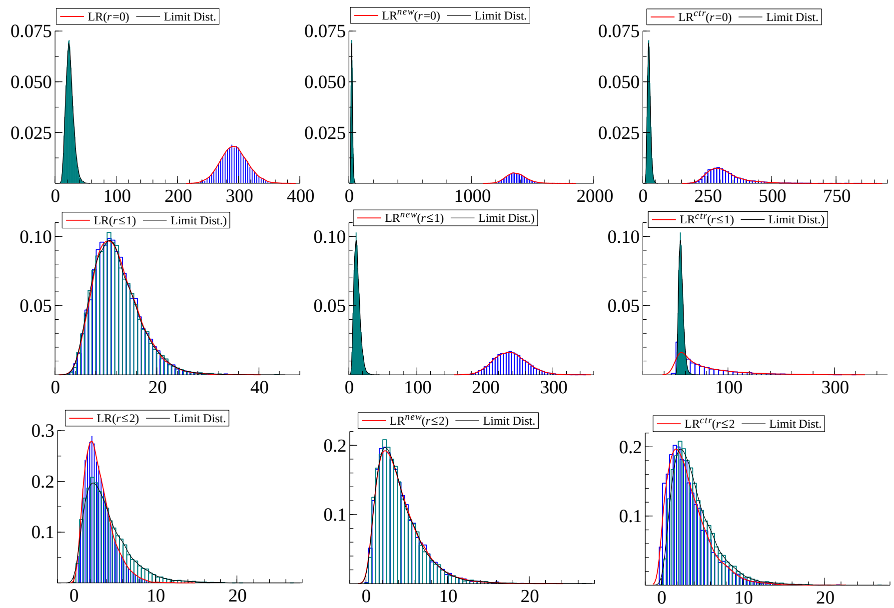

Figure 4: Distribution of cointegration test statistics (each row denotes a different null for r) at a sample of $T = 1,000$ observations in a 3-variate setting. The left column corresponds to the original data, the central one to the new data, and the column on the right to the controlled data.

we can avoid dealing with the *controlled* data. Our approach relies on a reinterpretation of the policy, in light of recent research on the topic.

The route we follow consists of a reinterpretation of the timing of the policy, using a framework delineated by Rambachan and Shephard (2021) and we borrow their explanations. In their approach to policy, at each period $t \geq 1$, the unobserved unit X_t receives a random assignment W_t and we observe an outcome $X_t^{new}(W_t)$. The "potential outcome" process at time t, for any deterministic sequence $\{w_s\}$, is $X_t^{new}\left(\{w_s\}_{s\geq 1}\right)$. Under the assumption of *Non-anticipating Potential Outcomes*, for each $t \geq 1$ and all deterministic sequences $\{w_t\}_{t\geq 1}$, $\{w_t'\}_{t\geq 1}$, the potential outcomes do not depend on future realisations:

$$X_t^{new}\left(w_{1:t}, \{w_s\}_{s\geq t+1}\right) \overset{a.s.}{=} X_t^{new}\left(w_{1:t}, \{w_s'\}_{s\geq t+1}\right).$$

Rambachan and Shephard make the link with the macroeconomic literature on impulse response functions (IRF), defined in the context of Structural VARs as (Sims *et al.*, 1982) for $h \geq 1$ as

$$\mathsf{IRF}_{k,t,h}\left(w_k, w_k'\right) \equiv \mathsf{E}\left[Y_{t+h}\left(W_{k,t}\right) \middle| W_{k,t} = w_k\right] - \mathsf{E}\left[Y_{t+h}\left(W_{k,t}\right) \middle| W_{k,t} = w_k'\right].$$

Rambachan and Shephard (2021) show that the IRF can be given a causal meaning, coinciding with the Average Treatment Effect $\mathsf{E}\left[Y_{t+h}\left(w_k\right) - Y_{t+h}\left(w_k'\right)\right]$ under some orthogonality conditions that are satisfied when the assignment constitutes a "shock", which they define as satisfying $W_{k,t} \perp \left(W_{1:t-1}, W_{k',t}, W_{t+1:t+h}, \left\{X_{t+h}^{new}\left(w_{1:t+h}\right)\right\}\right)$. From their definitions, the assignment W_t corresponding to the control policy is zero when the data generating process for X_t^{new} coincides with that of X_t. We can therefore define the assignment as the control in JJ:

$$W_{t+1} = \pi\nu\left(X_t^{new}\right). \tag{11}$$

for some matrix π to be defined. In our context where the policy is implemented at every period, Rambachan and Shephard (2021) define the *impulse causal effect at horizon* $h \geq 1$ as the difference between X_{t+h}^{new} and the *counterfactual* X_{t+h}^{*new} that obtains with the only change that the policy is not implemented at time t (so $W_t^* = \pi\nu\left(X_{t-1}^*\right) = 0$ under the counterfactual and this is the only difference in assignments between X_{t+h}^{new} and X_{t+h}^{*new}). In their words, the impulse causal effect measures the *ceteris paribus* causal effect – of intervening to switch the time-t assignment from 0 to W_t – on the h-period ahead outcomes, holding all else fixed along the assignment process. Since X_t is non-stationary in JJ, the impulse causal effect and its unconditional expectation, the Average Treatment Effect, vary with time. Yet, we notice that the policy intervention, W_{t+1} in (11) does not constitute a contemporaneous "shock" in the Ramey (2016) or Rambachan and Shephard (2021, Theorem 2) sense, since W_{t+1} is not unanticipated from, or uncorrelated with, lagged endogenous variables, in fact it is possibly persistent (though stationary under the assumption of controllability). In practice, JJ, Theorem 7, show there exists a linear policy rule which ensures that W_{t+1} can be expressed as a function of the lagged shocks to the unperturbed system and can be made *iid*. In the context of the VAR(1), following on the implementation of the policy, the data generating process writes

$$X_{t+1}^{new} = -\left[\alpha\mu + \left(I_p + \alpha\beta'\right)\overline{a}\kappa_0\right] + \left(I_p + \alpha\beta'\right)\left(I_p + \overline{a}\kappa'\right)X_t^{new} + \varepsilon_{t+1} \tag{12}$$

so $\kappa' X_{t+1}^{new} = -\kappa'\alpha\mu + \kappa'\left(I_p + \alpha\beta'\right)\left[\left(I_p + \overline{a}\kappa'\right)X_t^{new} - \overline{a}\kappa_0\right] + \kappa'\varepsilon_{t+1}$. Under policy assumptions $\kappa'\alpha = 0$ and $I_m + \kappa'\overline{a} = 0$. The previous expression then simplifies as

$$\kappa' X_{t+1}^{new} = \kappa'\left[\left(I_p + \overline{a}\kappa'\right)X_t^{new} - \overline{a}\kappa_0\right] + \kappa'\varepsilon_{t+1} = \kappa_0 + \kappa'\varepsilon_{t+1}$$

i.e., setting $\pi = \left(I_p + \alpha\beta'\right)$, we obtain $W_{t+1} = \pi\nu\left(X_t^{new}\right) = \left(I_p + \alpha\beta'\right)\overline{a}\kappa'\varepsilon_t$.

Hence, the data generating process under the new policy – the process for the potential outcome – becomes

$$\begin{aligned} \Delta X_{t+1}^{new} &= \alpha\left(\beta' X_t^{new} - \mu\right) + \left(I_p + \alpha\beta'\right)\overline{a}\kappa'\varepsilon_t + \varepsilon_{t+1}, \\ &= \alpha\left(\beta' X_t^{new} - \mu\right) + W_{t+1} + \varepsilon_{t+1} \end{aligned} \tag{13}$$

a vector autoregressive moving-average model, or a VARMA$(1,1)$ model. Using the results in Theorem 8 of JJ, we can show the same result for a VAR(k) model that becomes a

VARMA$(k,1)$ model under the policy. Hence, since $\varepsilon_t \perp \varepsilon_{t+1}$ holds, an alternative structural vector error (or equilibrium) correction (SVEC) representation is feasible

$$\Delta X_{t+1}^{new} = \alpha \left(\beta' X_t^{new} - \mu\right) + B \underline{\varepsilon}_{t+1} \tag{14}$$

where $\underline{\varepsilon}_{t+1} = \left(\varepsilon_{t+1}', \varepsilon_t'\right)'$ and B conforms with (13). In equation (14), $\underline{\varepsilon}_{t+1}$ contains an excess shock that is a priori recoverable from past observations (Chahrour and Jurado, 2022).

The analysis above shows that we can reinterpret the timing of the JJ framework through a standard SVEC (14): the authority does not exert a control at time t that modifies X_t into X_t^{ctr}. Instead, it observes X_t and introduces a direct shock to the system at time $t+1$ that relates to the observables at t. The intervention adds a shock W_{t+1} to the system that, as it does not correlate with ε_{t+1}, does not render the variance of the innovations singular.

Under the random assignment narrative, it is conceivable that the observed process should be X_t^{new} so under this interpretation, we avoid considering the controlled process as the only observable. In fact, it is not defined here. Yet the question remains of how the authorities manages to introduce this new shock to the system, a shock that shifts X_{t+1}^{new} without controlling it perfectly (as opposed to X_t^{ctr} in JJ). Intuitively, the natural route to achieving such a result relies on considering that the authority uses a primary tool that differs from the observable – partially controlled – policy instrument.

This setting is actually considered explicitly by JJ and we explore it further from the econometrician's perspective as a natural way to treat the issue of observables in the controlled VAR system. A key element in our discussion relies on equation (14), through which we see that we can retrieve empirically, through the VARMA structure the original (α, β) parameters (see Funovits, 2024), so the econometrician can ex post (that is, post policy implementation) perform the analysis that the policy maker does *ex ante*.

3 Policy with intermediate targets

In this section, we explore controlled policy further, with the intention of delineating how the authorities can put their policy in place. For this we focus on the situation where they cannot directly control the target through their instrument, but must rely on market forces through an intermediate target. This was considered in JJ but we show here how the applied econometrician can assess, and identify these targets. The choice of final and intermediate policy target variables as a problem encountered both by the policy maker, and then by the econometrician analysing time series data. We restrict ourselves to $m = 1$ to render the argument tractable *i.e.*, there is a singular policy target along with a singular policy instrument in the CVAR system. If the final target is recognised as $b'X_t$, the intermediate target is then stated as follows, according to JJ:

Definition 3.1 *The intermediate target variable $c'X_t$ for $c \in \mathbf{R}^p$ is a variable which is cointegrated with the final target variables $b'X_t$, so that there exists a stationary relationship $b'X_t + \phi c'X_t$ for $\phi \neq 0$.*

Applied economists usually find it possible to determine known vectors b and c along with a, guided by some prior knowledge on conceivable transmission mechanisms of economic policy. It is indeed straightforward to fix a, or to choose an instrument variable, on the basis of a policy tool available to monetary and fiscal authorities. However, selecting b and c in the context of an empirical study is considered more challenging, as either $b'X_t$ or $c'X_t$ may serve as final or intermediate targets. While insights from economic theory alone are generally insufficient to justify the selection process, a data-driven procedure becomes essential. The following subsections will discuss this approach in detail.

3.1 Identifying targets

Let us first consider a procedure for the empirical identification of the two types of policy targets. For this, we see that Definition 3.1 requires there exists $j \leq r$ such that

$$sp\left(\beta_j\right) \subset sp\left(b + \phi c\right) \quad \text{for} \quad \phi \neq 0,$$

where $sp\left(\cdot\right)$ denotes the vector space spanned by $\cdot$ and β_j is one of the cointegrating vectors in $\beta = (\beta_{1,...,}\beta_r)$. We then introduce below our definition of a purely-final policy target variable on the basis of the selection vector b.

Definition 3.2 *Suppose $sp\left(\beta_j\right) \subset sp\left(b + \phi c\right)$ holds with $\phi \neq 0$, along with $b = e_j$, where e_j denotes the j-th column vector of I_p subject to $j \leq r$. The selected variable $b'X_t$ is then defined as a purely-final policy target if it cannot act as a policy instrument for any other variables in the CVAR system, that is $Cb = 0$, and it reacts only to the cointegrating relationship $\beta_j' X_{t-1}$ among $\beta' X_{t-1}$ in the CVAR system.*

As an example, if $p = 4$, $r = 2$ and $j = 1$, it then follows that $b = (1, 0, 0, 0)$ and

$$C = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix},$$

so that $Cb = 0$, thereby allowing us to regard $b'X_t$ as not acting as a policy instrument for any other variables in the system. The zero column in the C matrix is guaranteed if

$$\alpha = (\alpha_1, \alpha_2) = \begin{pmatrix} \xi & * \\ 0 & * \\ 0 & * \\ 0 & * \end{pmatrix}$$

for a non-zero scalar parameter ξ, since

$$sp(\alpha_1) \subset sp(b)$$

is implied by the condition $Cb = \beta_\perp(\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp' b = 0$. Given this, an intermediate policy target cX_t needs to be incapable of playing the role of a purely-final policy target, so that

$$sp(\alpha_1) \not\subset sp(c)$$

should also be ensured.

We summarise the arguments above in a proposition, assuming that the policy only considers controls as unique variables, not linear combinations thereof.

Proposition 3.3 *Suppose $a = e_i$, $b = e_j$ and $c = e_k$ for $i \neq j \neq k$, $j \leq r$ and $i, k \leq p$, and the controllability condition $c'Ca \neq 0$ holds. The selected variable $b'X_t$ is identified as the purely-final policy target while $c'X_t$ as the intermediate policy target if the following three conditions are satisfied:*

1. $sp(\beta_j) \subset sp(b + \phi c) \quad for \quad \phi \neq 0,$
2. $sp(\alpha_j) \subset sp(b),$
3. $sp(\alpha_j) \not\subset sp(c).$

All the conditions here are empirically testable, so that we can treat this proposition as a pre-procedure for CVAR-based policy simulation exercises involving both intermediate and final policy targets. Section 4 below provides an empirical illustration of the procedure based on the proposition above.

As a corollary to this proposition, we present the following result:

Corollary 3.4 *Under the conditions of Proposition 3.3, the two vectors $b'C$ and $c'C$ are collinear, along with $b'Cb = c'Cb = 0$.*

Proof. See the Appendix. ■

This corollary has two interesting implications. First, it implies that $c'Ca \neq 0$ means $b'Ca \neq 0$ and *vice versa*, as indicated by JJ. Second, it can facilitate a SVEC-type analysis. In order to explain this second aspect, let us provide the Granger-Johansen representation in the context of the simplified model (2):

$$X_t = C \sum_{j=1}^{t} \varepsilon_i + \sum_{j=1}^{\infty} C_i^* \varepsilon_{t-i} + CX_0 - \alpha(\beta'\alpha)\mu,$$

where $\sum_{j=1}^{\infty} C_i^* \varepsilon_{t-i}$ represents a linear process with the matrices C_i^* decreasing exponentially fast. For the purpose of considering its structural interpretation in the context of SVEC formulation, we introduce a non-singular matrix G so that we can define $u_t = G\varepsilon_t$ for $G\Omega G' = I_p$ and find

$$X_t = \tilde{C} \sum_{j=1}^{t} u_i + \sum_{j=1}^{\infty} \tilde{C}_i^* u_{t-i} + CX_0 - \alpha \left(\beta' \alpha\right) \mu,$$

for $\tilde{C} = CG^{-1}$ and $\tilde{C}_i^* = C_i^* G^{-1}$. See Juselius (2006, Ch.15) *inter alia*, for further details of this type of formulation. The parameter $\tilde{C}$ represents the long-run impact matrix in this context and needs to be restricted to claim its structural interpretation. If we continue to use the example $p = 4$, $r = 2$ and $j = 1$, the corollary implies, as a result of matrix rotation,

$$\begin{aligned} \tilde{C} &= \begin{pmatrix} 0 & -\phi c_{22} & -\phi c_{23} & -\phi c_{24} \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} G^{-1} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \tilde{G}^{-1} \end{aligned} \tag{15}$$

where $\tilde{G}^{-1} = N^{-1} G^{-1}$ for

$$N = \begin{pmatrix} 1 & \phi \;\; 0 \;\; 0 \\ 0_{3\times 1} & I_3 \end{pmatrix}.$$

The presence of a zero row in (15) indicates that we can formulate a SVEC model in such a way that it is only the intermediate target that is influenced by a series of long-run structural shocks (permanent shocks), not the final target, and *vice versa*. This is a reflection of the long-run synchronisation of $b'X_t$ and $c'X_t$. The collinear structure reduces the parameters of $\tilde{C}$ and will be useful for its identification in the SVEC context.

3.2 New process in the classification of policy targets

The arguments presented in the above subsection imply that the derived new system can be re-expressed in a way that reveals the underlying structure resulting from the implementation of economic policy. The expression of the new system is provided in the next proposition, which is based on the simplified CVAR model (2) for the sake of simplicity.

Proposition 3.5 *Suppose that all the conditions in Proposition 3.3 are satisfied, so that $b'X_t$ and $c'X_t$ are identified as the purely-final policy target and the intermediate policy target,* *respectively. The system for X_t^{new} is then expressed as*

$$\Delta X_{t+1}^{new} = \alpha^\circ \left[(b, c, \delta)' X_t^{new} - \mu^\circ \right] + \varepsilon_{t+1}, \quad \textit{for} \quad t = k+1, ..., T, \tag{16}$$

where $(b, c, \delta) \in \mathbf{R}^{p \times (r+1)}$ represents a set of cointegrating vectors derived from the rotation of (κ, β) such that

$$(b, c, \delta)' X_t^{new} - \mu^\circ \sim \mathrm{I}(0)$$

and

$$sp(\delta) \subset sp(g_\perp)$$

for $g = (b, c)$, along with a set of adjustment vectors α° and constants μ° derived from $(a + \alpha\beta' a, \alpha)$ and $(\kappa^{*\prime}, \mu)'$ respectively, as a result of the rotation of (κ, β).

Proof. See the Appendix. ■

The derived system (16) explicitly shows that the selection vectors b and c are members of the cointegrating vectors for X_t^{new}, while the remaining cointegrating vectors consist of δ, which is orthogonal to b and c as a consequence of matrix rotation given b and c. The vectors $\delta' X_t^{new}$ are likely to contain the policy instrument $a' X_t^{new}$ as its constituent. This implies that the orthogonality of δ with respect to b and c can be interpreted as a representation of the X_t^{new}-based CVAR structure resulting from policy implementation in a counterfactual world. In other words, the cointegrating space for the original process is spanned by

$$\beta = [(b + \phi c)\,\omega, \beta_2] \in \mathbf{R}^{p \times r},$$

for a scalar ω, and this space is expanded to

$$\beta^\circ \equiv (b, c, \delta) \in \mathbf{R}^{p \times (r+1)}$$

for the new process, indicating that more stability has been attained in β° as result of the implementation of the policy in a counterfactual scenario. In the new process, not only $b' X_t^{new}$ but also $c' X_t^{new}$ is individually a stationary series, while $\delta' X_t^{new}$, presumably including $a' X_t^{new}$ as its constituent, is now seen as having no interactions with the two stationary series as a result of policy implementation. The cointegrating vectors for the new system will be estimated in the empirical illustration below to confirm the argument presented in this subsection.

It is also possible to re-express (16) within the context of the SVEC-based reinterpretation developed in Section 2.4. Recalling the identity $C = I_p - \alpha(\beta'\alpha)^{-1}\beta'$, along with the structure of $\beta = [(b + \phi c)\,\omega, \beta_2]$, we can manipulate κ' to yield

$$\kappa' = -(b' C \overline{a})^{-1} b' C = \zeta b' + \vartheta \phi c' + \vartheta \beta_2',$$

where $\vartheta = (b'C\overline{a})^{-1} b'\alpha \left(\beta'\alpha\right)^{-1} \omega$, and $\zeta = -(b'C\overline{a})^{-1} + \vartheta$. The SVEC representation is

$$\Delta X_{t+1}^{new} = \alpha \left(\beta' X_t^{new} - \mu\right) + B\underline{\varepsilon}_{t+1}, \tag{17}$$

where B is now given as

$$B = \left[I_p, \left(I_p + \alpha\beta'\right) \overline{a} \left(\zeta, \vartheta\phi, \vartheta\right) \left(b, c, \beta_2\right)'\right],$$

so that we can interpret the random assignment $W_{t+1} = \left(I_p + \alpha\beta'\right) \overline{a} \left(\zeta, \vartheta\phi, \vartheta\right) \left(b, c, \beta_2\right)' \varepsilon_t$ as being partially driven by deviations from intermediate as well as final target values.

4 Empirical application

In this section we provide an empirical application of the above propositions to a series of macroeconomic data of New Zealand. We begin by examining the cointegrating rank of an empirical VAR system and then applies the suggested procedure to the data to distinguish between intermediate and final policy targets in the context of inflation targeting. We also conduct policy simulation exercises using the empirical CVAR system.

4.1 Cointegrated VAR

We start with the estimation of an unrestricted VAR model for X_t consisting of New Zealand's quarterly macroeconomic series:

$$X_t = \left(\pi_t, \pi_t^e, y_t, i_t\right)',$$

where π_t is a realised annual (year-on-year) inflation rate, π_t^e is a survey-based annual inflation expectation, y_t is the log of real output, i_t is the short-term interest rate. Further details of the data are provided in the Appendix. The estimation period spans from the third quarter of 1992 to the first quarter of 2020, encompassing a total of 111 observations. The endpoint was chosen to account for the significant impact of the COVID-19 pandemic on New Zealand's economy beginning in 2020 and continuing thereafter. Figure 5 presents an overview of the data for the four variables. All the series appear to be non-stationary; notably, π_t and π_t^e have exhibited synchronised movements, accompanied by a clear upward trend in y_t. We thus deem it suitable to employ a trend-restricted I(1) CVAR method for the analysis of the data.

According to a preliminary regression analysis some of the lagged dynamic terms at $k = 4$ are judged to be fairly significant, resulting in the selection of a VAR(4) model for further study. Figure 6 displays a battery of diagnostic graphs calculated from the estimated VAR(4) model: scaled residuals (the first column), residual autocorrelation functions

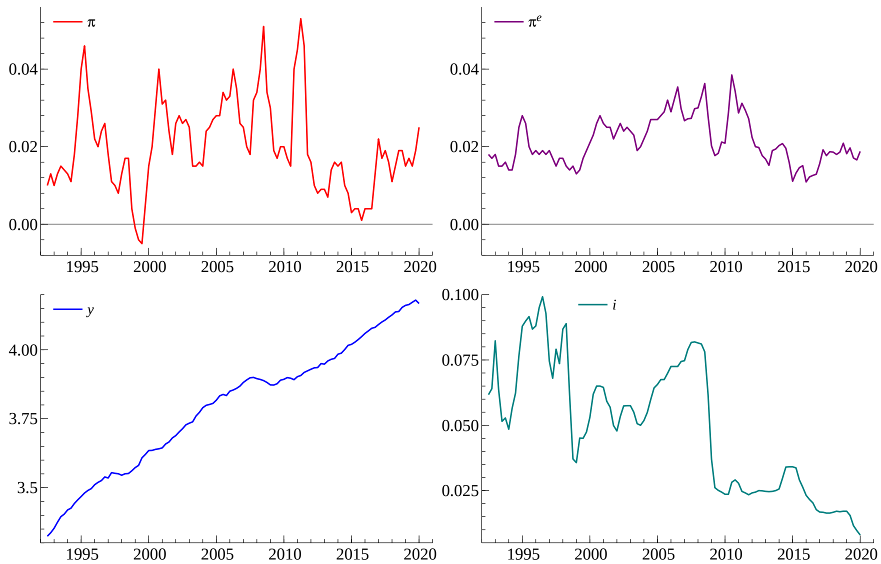

Figure 5: An overview of the data

Table 1: Inference on the cointegrating rank for X_t.

	$r = 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
log LR	84.738[0.000]**	45.168[0.027]*	21.292[0.169]	6.815[0.375]

Note. Figures in square brackets are p-values.

** and * denote significance at the 1% and the 5% level, respectively.

(ACF, the second column) and residual quantile-quantile plots against normality (QQ plot, the third column). The residuals appear to be free from serial correlations, providing evidence in support of a quasi likelihood-based analysis of cointegration studied by Kurita and Nielsen (2019). The cointegration literature also shows that trace tests for the selection of cointegrating rank are robust to non-normality in the innovation term; see Cheung and Lai (1993), *inter alia*, for further details. The evidence recorded in the figure thus justifies using the VAR(4) model as a basis for exploring the underlying cointegrating rank.

Table 1 reports a class of trace test statistics for the choice of r, $\log LR(r|p)$ for $r = 0, ..., 3$ given $p = 4$. The series of tests are in support of $r = 2$ at the 5% level, so we select this value as the retained cointegration rank. We then proceed to applying the procedure based on Proposition 3.3, for which we recall that likelihood ratio tests for restrictions on α and β have asymptotic χ^2 distributions, given the selection of cointegrating rank (see Johansen, 1996, Chs. 7 and 8 for further details).

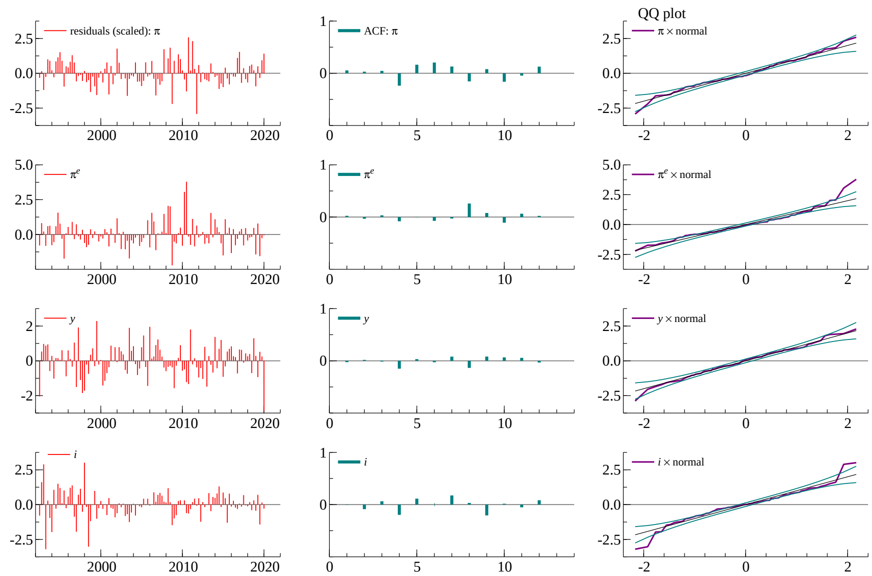

Figure 6: Residual diagnostics

4.2 Classifying the policy targets

The hypothetical long-run structure we envision is as follows: (i) expected inflation serves as the intermediate target while actual inflation is the final target, resulting in the long-run synchronisation of the two inflation rates, and (ii) expected inflation is driven by the output gap and interest rate, leading to a long-run Phillips curve formulation. Given this hypothetical structure as well as the selection of the interest rate as an instrument variable, we conceive the specification of $a = (0, 0, 0, 1)'$, $b = (1, 0, 0, 0)'$, $c = (0, 1, 0, 0)'$ and $\phi = -1$; that is, $a'X_t = i_t$, $b'X_t = \pi_t$, $c'X_t = \pi_t^e$ and $(b - c)X_t = \pi_t - \pi_t^e$. The parameter ϕ can be estimated in the CVAR framework but it seems natural to preset $\phi = -1$ as a hypothesis, suggesting a presumed synchronisation of π_t and π_t^e. This specification then allows us to test for the validity of the three hypotheses given in Proposition 3.3, according to which we should fail to reject the first two and reject the third. The first hypothesis to be tested under the above specification is

$$\mathrm{H}_0^{(1)} : sp\,(\beta_1) \subset sp\,(b - c)\,. \tag{18}$$

In order to identify the cointegrating space, we have also introduced a normalisation scheme for the second cointegrating vector (0 and 1 for the first and the second element, respectively), arriving at the following estimates:

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix}=\begin{bmatrix}\underset{(0.097)}{-0.449} & \underset{(0.101)}{-0.152}\\ \underset{(0.043)}{0.013} & \underset{(0.045)}{-0.218}\\ \underset{(0.120)}{-0.062} & \underset{(0.125)}{-0.179}\\ \underset{(0.095)}{0.003} & \underset{(0.099)}{-0.147}\end{bmatrix}\begin{bmatrix}\underset{(-)}{1} & \underset{(-)}{0}\\ \underset{(-)}{-1} & \underset{(-)}{1}\\ \underset{(-)}{0} & \underset{(0.045)}{-0.240}\\ \underset{(-)}{0} & \underset{(0.116)}{0.494}\\ \underset{(-)}{0} & \underset{(0.0004)}{0.002}\end{bmatrix}'\begin{pmatrix}\pi_{t-1}\\ \pi^e_{t-1}\\ y_{t-1}\\ i_{t-1}\\ t\end{pmatrix}. \tag{19}$$

The log-likelihood ratio test statistic ($\log LR$) is 4.046[0.257] with its p-value, according to $\chi^2(3)$, given in the square brackets, so that $\mathrm{H}_0^{(1)}$ is not rejected at conventional levels.

Next is the testing of

$$\mathrm{H}_0^{(2)} : sp\,(\alpha_1) \subset sp\,(b)\,, \tag{20}$$

under $\mathrm{H}_0^{(1)}$. Imposing a set of additional restrictions consistent with $\mathrm{H}_0^{(2)}$ yields

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix}=\begin{bmatrix}\underset{(0.092)}{-0.462} & \underset{(0.099)}{-0.146}\\ \underset{(-)}{0} & \underset{(0.043)}{-0.210}\\ \underset{(-)}{0} & \underset{(0.118)}{-0.202}\\ \underset{(-)}{0} & \underset{(0.094)}{-0.148}\end{bmatrix}\begin{bmatrix}\underset{(-)}{1} & \underset{(-)}{0}\\ \underset{(-)}{-1} & \underset{(-)}{1}\\ \underset{(-)}{0} & \underset{(0.045)}{-0.241}\\ \underset{(-)}{0} & \underset{(0.118)}{0.508}\\ \underset{(-)}{0} & \underset{(0.0003)}{0.002}\end{bmatrix}'\begin{pmatrix}\pi_{t-1}\\ \pi^e_{t-1}\\ y_{t-1}\\ i_{t-1}\\ t\end{pmatrix}, \tag{21}$$

along with $\log LR = 4.395[0.623]$ on the basis of $\chi^2\,(6)$, hence leading to the conclusion that $\mathrm{H}_0^{(2)}$ fails to be rejected.

As the third step, we are going to check the *rejection* of

$$\mathrm{H}_0^{(3)} : sp\,(\alpha_1) \subset sp\,(c)$$

under $\mathrm{H}_0^{(1)}$, so that $sp\,(\alpha_1) \not\subset sp\,(c)$ holds. The resulting estimates are given below:

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix}=\begin{bmatrix}\underset{(-)}{0} & \underset{(0.115)}{-0.312}\\ \underset{(0.041)}{0.081} & \underset{(0.049)}{-0.254}\\ \underset{(-)}{0} & \underset{(0.128)}{-0.233}\\ \underset{(-)}{0} & \underset{(0.102)}{-0.137}\end{bmatrix}\begin{bmatrix}\underset{(-)}{1} & \underset{(-)}{0}\\ \underset{(-)}{-1} & \underset{(-)}{1}\\ \underset{(-)}{0} & \underset{(0.038)}{-0.199}\\ \underset{(-)}{0} & \underset{(0.098)}{0.433}\\ \underset{(-)}{0} & \underset{(0.0003)}{0.002}\end{bmatrix}'\begin{pmatrix}\pi_{t-1}\\ \pi^e_{t-1}\\ y_{t-1}\\ i^s_{t-1}\\ t\end{pmatrix}.$$

The corresponding test statistic is $\log LR = 26.846[0.0002]^{**}$ according to $\chi^2\,(6)$, strong evidence against $\mathrm{H}_0^{(3)}$, so we are able to reject this hypothesis. Overall, we conclude that

$b'X_t = \pi_t$ is identified as the purely-final policy target while $c'X_t = \pi_t^e$ as the intermediate policy target.

Finally, getting back to (21), we introduce a zero restriction on the first element of the second adjustment vector, so the feedback mechanism is consistent with the identification scheme for the cointegrating space:

$$\hat{\alpha}\begin{pmatrix}\hat{\beta}\\ \hat{\rho}\end{pmatrix}'\begin{pmatrix}X_{t-1}\\ t\end{pmatrix} = \begin{bmatrix} \underset{(0.089)}{-0.502} & \underset{(-)}{0} \\ \underset{(-)}{0} & \underset{(0.0411)}{-0.193} \\ \underset{(-)}{0} & \underset{(0.118)}{-0.238} \\ \underset{(-)}{0} & \underset{(0.094)}{-0.139} \end{bmatrix} \begin{bmatrix} \underset{(-)}{1} & \underset{(-)}{0} \\ \underset{(-)}{-1} & \underset{(-)}{1} \\ \underset{(-)}{0} & \underset{(0.046)}{-0.224} \\ \underset{(-)}{0} & \underset{(0.119)}{0.497} \\ \underset{(-)}{0} & \underset{(0.0004)}{0.002} \end{bmatrix}' \begin{pmatrix}\pi_{t-1}\\ \pi_{t-1}^e\\ y_{t-1}\\ i_{t-1}\\ t\end{pmatrix}, \tag{22}$$

along with $\log LR = 6.575[0.474]$, hence $\mathrm{H}_0^{(2)}$ being non-rejected according to $\chi^2(7)$. The identified structure in (22) indicate clearly how the instrument affects the intermediate target, thus having an influence on the purely-final policy target.

The condition for the controllability of $c'X_t = \pi_t^e$ by means of $a'X_t = i_t$ is given as $c'Ca \neq 0$, which implies $b'Ca = c'Ca \neq 0$; see Corollary 3.4. The parameter estimates $\hat{\alpha}$ and $(\hat{\beta}', \hat{\rho}')$ recorded in (22) have been used in the estimation of the C matrix:

$$\hat{C} = \begin{bmatrix} \underset{(-)}{0} & \underset{(-)}{0.206} & \underset{(-)}{0.038} & \underset{(-)}{-0.351} \\ \underset{(-)}{0} & \underset{(0.134)}{0.206} & \underset{(0.057)}{0.038} & \underset{(\mathbf{0.072})}{\mathbf{-0.351}} \\ \underset{(-)}{0} & \underset{(0.868)}{-1.695} & \underset{(0.370)}{1.525} & \underset{(0.467)}{-0.260} \\ \underset{(-)}{0} & \underset{(0.445)}{1.179} & \underset{(0.190)}{0.612} & \underset{(0.239)}{0.588} \end{bmatrix},$$

in which figures in parentheses denote standard errors. Inference concerning $\hat{C}$ is made on the basis of Paruolo (1997). The element $\hat{C}_{24}$ in bold corresponds to $c'\hat{C}a$, which is judged to be significantly different from 0 at the conventional significance level. The first and second rows of $\hat{C}$ (that is, $b'C$ and $c'C$) are identical along with the first column zero, aligned with Corollary 3.4, covering the identity $c'\hat{C}a = \hat{C}_{24} = \hat{C}_{14} = b'\hat{C}a$. With the aim of illuminating the roles of π_t^e and π_t in $X_t = (\pi_t, \pi_t^e, y_t, i_t)'$, we refer to Corollary 3.4 and rotate the above matrix to find

$$\hat{C}N = \begin{pmatrix} \mathbf{0} & 0 & 0 & 0 \\ \mathbf{0} & 0.206 & 0.038 & \mathbf{-0.351} \\ \mathbf{0} & -1.695 & 1.525 & -0.260 \\ \mathbf{0} & 1.179 & 0.612 & 0.588 \end{pmatrix}$$

for

$$N = \begin{pmatrix} \mathbf{1} & \mathbf{-1} \;\; \mathbf{0} \;\; \mathbf{0} \\ 0_{3\times 1} & I_3 \end{pmatrix}.$$

A series of zeros in the first column of $\hat{C}N$ indicates π_t lacks the capability to influence all the other variables in the system, thus categorising π_t as being subject to the other series in the context of policy control, while $c'\hat{C}Na \neq 0$ indicates π_t^e can be controlled by i_t; furthermore, the structure of the first row of N indicates that π_t and π_t^e form a stationary linear combination. We are thus justified in the conclusion that the instrument $a'X_t = i_t$ is employed to control the intermediate target $c'X_t = \pi_t^e$, which is cointegrated with the final target $b'X_t = \pi_t$, so that the final target is also controllable by the instrument by way of the intermediate target.

4.3 Empirical analysis of a simulated policy

We now employ the empirical CVAR derived above to simulate a class of new processes X_t^{new} subject to the control rule. The results in the previous subsection allow us to focus on controlling of $c'X_t = \pi_t^e$ by means of $a'X_t = i_t$. We therefore conduct a simulation exercise controlling inflation expectation rather than actual inflation. We conduct a set of two simulation studies here to confirm the theoretical arguments presented in the previous section.

First, we derive X_t^{new} with the target value $c^* = 0.015$, which is deliberately set lower than 0.02 (that is, 2%), the actual target rate adopted by the RBNZ, to illustrate how the policy simulation works. Figure 7(a) records the new instrument $a'X_t^{new} = i_t^{new}$ under the projected policy, together with the actual $a'X_t = i_t$; the former tends to move above the latter, indicating the responses of tighter monetary policy attaining the counterfactual target value $c^* = 0.015$. This monetary contraction has caused $c'X_t^{new} = \pi_t^{e,new}$ to move around the target level in Figure 7(b), where the new series appears to be stationary about a mean of $c^* = 0.015$, in contrast to the actual $c'X_t = \pi_t^e$, which exhibits more a clearly non-stationary behaviour than $c'X_t^{new} = \pi_t^{e,new}$. We can therefore conclude that expected inflation can be manipulated to achieve its pre-specified target level in the counterfactual world by means of the short-term rate instrument, as expected from $c'\hat{C}a < 0$ discussed in the previous subsection. This also leads us to argue that actual inflation can also be controlled owing to its synchronisation to the expected inflation rate.

Second, we put ourselves in the position of the econometrician in this counterfactual policy environment, and ask: would her empirical analysis be able to detect the policy? To answer this, we conduct a cointegration study of X_t^{new} derived from the above simulation to verify Proposition 3.5. Table 2 reports, in the same manner as Table 1, a set of trace test statistics calculated from the generated X_t^{new} series. The results provide strong evidence supporting $r = 3$, as predicted, in contrast to $r = 2$ recorded in Table 1. The selection of $r = 3$ enables us to further explore whether a cointegrating structure consistent with Proposition 3.5 truly underlies the system for X_t^{new}. The revealed cointegrating relationships

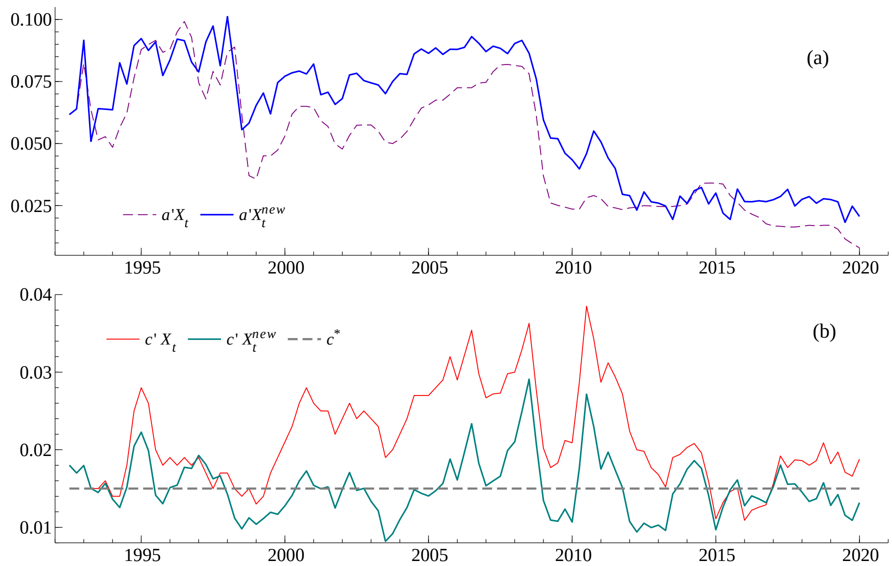

Figure 7: Policy simulation

Table 2: Inference on the cointegrating rank for X_t^{new}.

	$r = 0$	$r \leq 1$	$r \leq 2$	$r \leq 3$
$\log\ LR$	128.05[0.000]**	63.861[0.000]**	33.007[0.000]**	9.118[0.178]

Note. Figures in square brackets are p-values.

** denotes significance at the 1% level.

(apart from the linear trend) are

$$(b, c, \delta)' X_t^{new} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \underset{(0059)}{-2.223} \end{bmatrix}' \begin{pmatrix} \pi_{t-1}^{new} \\ \pi_{t-1}^{e,new} \\ y_{t-1}^{new} \\ i_{t-1}^{new} \end{pmatrix},$$

and the test statistic is $\log LR = 4.20[0.380]$ according to $\chi^2(4)$, thus accepting the null of the joint restrictions. The revealed structure aligns with the predictions in Proposition 3.5, representing a counterfactual world where both actual and expected inflation rates have become stationary as a consequence of a series of policy interventions. This is accompanied by the stationary combination of y_{t-1}^{new} and i_{t-1}^{new} alone (not including $\pi_{t-1}^{e,new}$), which contrasts with the second cointegrating relationship in (22) that consists of π_{t-1}^{e}, y_{t-1} and i_{t-1}.

5 Conclusion

This paper has explored the application of CVAR-based control theory from an econometrician's perspective, focusing on model estimation and counterfactual analysis. By reexamining the mechanisms underlying JJ's theoretical results, we have discussed some of their possible shortcomings from the perspective of the applied econometrician. Monte Carlo studies have illustrated the underlying statistical properties of new and controlled processes, reinforcing the argument that inference based on the controlled process is unreliable. We have also shown that the same mathematical results can be interpreted through a different timing of policy. Here the timing of the JJ framework can be reinterpreted in the context of SVEC. Instead of modifying policy variables contemporaneously, the authority introduces a structural shock that is entirely recoverable from past observations, thus avoiding singularity in the variance of innovations. In this random assignment interpretation the observed process should be the *new* process. In addition, this paper has presented a data-driven procedure for categorising intermediate and final policy targets within a model framework. The effectiveness of this procedure has been demonstrated through an analysis of New Zealand's monetary policy data. This paper hopes to lay the basis for future research aimed at enhancing the practicality of CVAR-based control analysis in applied macroeconomic and financial studies.

Acknowledgement:

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Appendix

A Proofs referring to Sections 2 and 3:

Derivation of the controlled process in Section 2.2.2:

The control rule is given by

$$\kappa' = -\left(b'C\overline{a}\right)^{-1} b'C \quad and \quad \kappa_0 = -\left(b'C\overline{a}\right)^{-1} \left[b'\pi^* - b'\alpha\left(\beta'\alpha\right)^{-1}\mu\right], \tag{23}$$

so by construction

$$\begin{aligned} \kappa'\overline{a} &= -\left(b'C\overline{a}\right)^{-1} b'C\overline{a} = -I_m, \\ I_m + \kappa'\overline{a} &= 0_m. \end{aligned}$$

Now, consider $x_{t+1}^{ctr} = \left(I_p + \overline{a}\kappa'\right) x_{t+1}^{new} - \overline{a}\kappa_0$ with $\kappa'\alpha = 0$ and $I_m + \kappa'\overline{a} = 0$. Using the dynamics of the 'new' process (see equation (23), p. 31 in JJ), namely,

$$\Delta x_{t+1}^{new} = \left[\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right] \begin{bmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa_0 \end{bmatrix} + \varepsilon_{t+1},$$

we see that

$$\Delta x_{t+1}^{ctr} = \left(I_p + \overline{a}\kappa'\right)\Delta x_{t+1}^{new} = \left(I_p + \overline{a}\kappa'\right)\left[\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right] \begin{bmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa_0 \end{bmatrix} + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1},$$

where $\left(I_p + \overline{a}\kappa'\right)\alpha = \alpha$, and

$$\begin{aligned} \left(I_p + \overline{a}\kappa'\right)\left(I_p + \alpha\beta'\right)\overline{a} &= \left(I_p + \alpha\beta' + \overline{a}\kappa'\right)\overline{a}, \\ &= \left(I_p + \alpha\beta'\right)\overline{a} - \overline{a}. \end{aligned}$$

Hence,

$$\left(I_p + \overline{a}\kappa'\right)\left(\alpha, \left(I_p + \alpha\beta'\right)\overline{a}\right) = \alpha\left(I_r, \beta'\overline{a}\right).$$

The controlled process therefore becomes

$$\begin{aligned} \Delta x_{t+1}^{ctr} &= \alpha\left(I_r, \beta'\overline{a}\right) \begin{bmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa_0 \end{bmatrix} + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ &= \alpha\left[\beta' x_t^{new} - \mu + \beta'\overline{a}\left(\kappa' x_t^{new} - \kappa_0\right)\right] + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ &= \alpha\left[\beta'\left(I_p + \overline{a}\kappa'\right) x_t^{new} - \mu\right] + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \\ \Delta x_{t+1}^{ctr} &= \alpha\left(\beta' x_t^{ctr} - \mu\right) + \left(I_p + \overline{a}\kappa'\right)\varepsilon_{t+1} \end{aligned}$$

i.e., although the controlled process seems to exhibit one extra cointegration relation, the latter is by construction proportional to the original one so it is degenerate. In other words, the controlled process can be expressed as

$$\Delta x_{t+1}^{ctr} = \alpha \left(I_r, \overline{a} \right) \left[\begin{array}{c} \beta' x_t^{ctr} - \mu \\ \kappa' x_t^{ctr} - \kappa_0 \end{array} \right] + \left(I_p + \overline{a} \kappa' \right) \varepsilon_{t+1},$$

and pre-multiplying this expression by κ' and performing further manipulation yields the following identity:

$$\begin{aligned} \kappa' x_{t+1}^{ctr} &= \kappa' x_t^{ctr} + \kappa' \overline{a} \left(\kappa' x_t^{ctr} - \kappa_0 \right) + \kappa' \left(I_p + \overline{a} \kappa' \right) \varepsilon_{t+1} \\ &= \left(I_p + \kappa' \overline{a} \right) \kappa' x_t^{ctr} - \kappa' \overline{a} \kappa_0 + \left(I_p + \kappa' \overline{a} \right) \kappa' \varepsilon_{t+1} \\ &= \kappa_0, \end{aligned}$$

due to $\kappa' \left(I_p + \overline{a} \kappa' \right) = \left(I_p + \kappa' \overline{a} \right) \kappa' = 0$. It then follows that $\kappa' \Delta x_{t+1}^{ctr} = 0$. The matrix $\left(I_p + \overline{a} \kappa' \right) \Omega \left(I_p + \overline{a} \kappa' \right)'$ is singular since

$$\left(I_p + \overline{a} \kappa' \right) \Omega \left(I_p + \overline{a} \kappa' \right)' \kappa = 0,$$

the matrix has a zero eigenvalue and its determinant is zero.

Next, we extend the above to a CVAR system with $k > 1$. Without loss of generality, we fix $k = 3$ and provide the system in companion form by following JJ:

$$\Delta \tilde{X}_t = \tilde{\alpha} \left(\tilde{\beta}' \tilde{X}_{t-1} - \tilde{\mu} \right) + \tilde{\varepsilon}_t,$$

where

$$\tilde{X}_t = \begin{pmatrix} X_t \\ X_{t-1} \\ X_{t-2} \end{pmatrix}, \quad \tilde{\varepsilon}_t = \begin{pmatrix} \varepsilon_t \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{\mu} = \begin{pmatrix} \mu \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \Gamma_1 & \Gamma_2 \\ 0 & I_p & 0 \\ 0 & 0 & I_p \end{pmatrix} \quad \text{and} \quad \tilde{\beta} = \begin{pmatrix} \beta & I_p & 0 \\ 0 & -I_p & I_p \\ 0 & 0 & -I_p \end{pmatrix},$$

so that we find

$$\tilde{\alpha}_\perp = \left(\alpha_\perp', -\alpha_\perp' \Gamma_1, -\alpha_\perp' \Gamma_2 \right)' \quad \text{and} \quad \tilde{\beta}_\perp = \left(\beta_\perp', \beta_\perp', \beta_\perp' \right)'.$$

In addition, recalling the definition $C = \beta_\perp (\alpha_\perp' \Gamma \beta_\perp)^{-1} \alpha_\perp'$ when $k > 1$, we introduce

$$\tilde{a} = \left(a', 0, 0 \right)' \quad \tilde{b} = \left(b', 0, 0 \right)' \quad \text{and} \quad \tilde{\kappa} = \left(\kappa_1', \kappa_2', \kappa_3' \right)',$$

for $\kappa_1 = -\left(b' C \overline{a} \right)^{-1} b' C$, $\kappa_2 = -\Gamma_1' \kappa_1$ and $\kappa_3 = -\Gamma_2' \kappa_1$ so that $\tilde{\kappa}' \tilde{\alpha} = 0$ holds. Define

$$\tilde{X}_t^{new} = \left(X_t^{new\prime}, X_{t-1}^{ctr\prime}, X_{t-2}^{ctr\prime} \right)' \quad \text{and} \quad \tilde{X}_t^{ctr} = \left(X_t^{ctr\prime}, X_{t-1}^{ctr\prime}, X_{t-2}^{ctr\prime} \right)',$$

which are driven by

$$\tilde{X}_t^{new} = \left(I_{3p} + \tilde{\alpha}\tilde{\beta}'\right)\tilde{X}_{t-1}^{ctr} - \tilde{\alpha}\tilde{\mu} + \tilde{\varepsilon}_t, \tag{24}$$

$$\tilde{X}_t^{ctr} = \tilde{X}_t^{new} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\left(\tilde{\kappa}'\tilde{X}_t^{new} - \kappa_0\right), \tag{25}$$

for $\kappa_0 = -(b'C\overline{a})^{-1}[b^* - b'\left(I_p - C\Gamma\right)\overline{\beta}\mu]$. Substituting (24) into (25) leads to

$$\begin{aligned}\tilde{X}_t^{ctr} &= \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\left[\left(I_{3p} + \tilde{\alpha}\tilde{\beta}'\right)\tilde{X}_{t-1}^{ctr} - \tilde{\alpha}\tilde{\mu} + \tilde{\varepsilon}_t\right] - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa_0 \\ &= \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\left(I_{3p} + \tilde{\alpha}\tilde{\beta}'\right)\tilde{X}_{t-1}^{ctr} - \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\alpha}\tilde{\mu} \\ &\qquad - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa_0 + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_t.\end{aligned}$$

This is reduced to, due to $\tilde{\kappa}'\tilde{\alpha} = 0$,

$$\tilde{X}_t^{ctr} = \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}' + \tilde{\alpha}\tilde{\beta}'\right]\tilde{X}_{t-1}^{ctr} - \tilde{\alpha}\tilde{\mu} - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa_0 + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_t.$$

Hence, we arrive at

$$\begin{aligned}\Delta\tilde{X}_t^{ctr} &= \left[\tilde{\alpha}\tilde{\beta}' + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{X}_{t-1}^{ctr} - \tilde{\alpha}\tilde{\mu} - \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa_0 + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_t \\ &= \left[\tilde{\alpha}\tilde{\beta}', \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\right]\left[\begin{array}{c}\tilde{\beta}'\tilde{X}_{t-1}^{ctr} - \tilde{\mu} \\ \tilde{\kappa}'\tilde{X}_{t-1}^{ctr} - \kappa_0\end{array}\right] + \left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_t.\end{aligned}$$

Noting that $I_m + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1} = I_m + \kappa_1' a\left(a'a\right)^{-1} = 0$, we pre-multiply the above equation by $\tilde{\kappa}'$ to derive the identity

$$\begin{aligned}\tilde{\kappa}'\tilde{X}_t^{ctr} &= \tilde{\kappa}'\tilde{X}_{t-1}^{ctr} + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\left(\tilde{\kappa}'\tilde{X}_{t-1}^{ctr} - \kappa_0\right) + \tilde{\kappa}'\left[I_{3p} + \tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\tilde{\kappa}'\right]\tilde{\varepsilon}_t \\ &= \left[I_m + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\right]\tilde{\kappa}'\tilde{X}_{t-1}^{ctr} - \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\kappa_0 + \left[I_m + \tilde{\kappa}'\tilde{a}\left(\tilde{a}'\tilde{a}\right)^{-1}\right]\tilde{\kappa}'\tilde{\varepsilon}_t \\ &= \kappa_0.\end{aligned} \tag{26}$$

Hence,

$$\tilde{\kappa}'\tilde{X}_t^{ctr} = \kappa_1'\left(X_t^{ctr} - \Gamma_1 X_{t-1}^{ctr} - \Gamma_2 X_{t-2}^{ctr}\right) = \kappa_0,$$

or, equivalently, its general expression is

$$\kappa_1'\Gamma\left(L\right)X_t^{ctr} = \kappa_0, \tag{27}$$

for $\Gamma\left(L\right) = I_p - \Gamma_1 L - \cdots - \Gamma_k L^k$. ■

Proof of Corollary 3.4:

Referring to the definition of the C matrix, we find

$$\left(b' + \phi c'\right)C = \left(b' + \phi c'\right)\beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp' = 0,$$

which yields the collinearity $b'C = -\phi c'C$ for $\phi \neq 0$. The result $b'Cb = c'Cb = 0$ follows directly from $Cb = 0$. ■

Proof of Proposition 3.5:

The new process for $t = k+1, ..., T$ is expressed as

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha) \left[\begin{pmatrix} \kappa' \\ \beta' \end{pmatrix} X_{t+1}^{new} - \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} \right] + \varepsilon_{t+1}.$$

Recall the set of known vectors: $a = e_i$, $b = e_j$ and $c = e_k$ for $i \neq j \neq k$, $j \leq r$ and $i, k \leq p$. Noting the identity $\kappa' = -(b'C\overline{a})^{-1} b'C = -(b'C\overline{a})^{-1} b' + (b'C\overline{a})^{-1} b'\alpha \left(\beta'\alpha\right)^{-1} \beta'$, we can rewrite the above as

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha) \, R_1 \left[\begin{pmatrix} b' \\ \beta' \end{pmatrix} X_{t+1}^{new} - R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} \right] + \varepsilon_{t+1},$$

where R_1 is a rotation matrix defined as

$$R_1 = \begin{pmatrix} -(b'Ca)^{-1} & (b'C\overline{a})^{-1} b'\alpha \left(\beta'\alpha\right)^{-1} \\ 0 & I \end{pmatrix},$$

and the constant term is subject to

$$e_j' R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} = b^*.$$

On the basis of $sp\,(\beta) \subset sp\,(b + \phi c)$ for $\phi \neq 0$, we split $\beta = (\beta_1, \beta_2) = [(b + \phi c)\,\omega, \beta_2]$ for a non-zero scalar ω, where $\beta_2 \in \mathbf{R}^{p\times(r-1)}$ along with a class of conformable decompositions of α and the constant term:

$$\alpha = (\alpha_1, \alpha_2) \quad \text{and} \quad R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix} = \begin{pmatrix} b^* \\ \mu_1 \\ \mu_2 \end{pmatrix}.$$

These decompositions lead to

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha_1, \alpha_2) \, R_1 \left[\begin{pmatrix} b' \\ \omega\,(b' + \phi c') \\ \beta_2' \end{pmatrix} X_{t+1}^{new} - \begin{pmatrix} b^* \\ \mu_1 \\ \mu_2 \end{pmatrix} \right] + \varepsilon_{t+1}. \tag{28}$$

Furthermore, by noting that b and c are mutually orthogonal unit vectors, we introduce another rotation matrix

$$R_2 = \begin{pmatrix} 1 & 0 & 0_{1\times(r-1)} \\ 0 & 1 & 0 \\ b'\beta_2 e_1^* & c'\beta_2 e_1^* & \\ \vdots & \vdots & I_{r-1} \\ b'\beta_2 e_{r-1}^* & c'\beta_2 e_{r-1}^* & \end{pmatrix} \begin{pmatrix} 1 & 0 & 0_{1\times(r-1)} \\ \omega & \omega\phi & 0 \\ 0 & 0 & \\ \vdots & \vdots & I_{r-1} \\ 0 & 0 & \end{pmatrix},$$

where e_j^* denotes the j-th column vector of I_{r-1}. Applying R_2 to (28), we derive

$$\Delta X_{t+1}^{new} = (a + \alpha\beta' a, \alpha_1, \alpha_2)\, R_1 R_2 \left[\begin{pmatrix} b' \\ c' \\ \delta' \end{pmatrix} X_{t+1}^{new} - R_2^{-1} \begin{pmatrix} b^* \\ \mu_1 \\ \mu_2 \end{pmatrix} \right] + \varepsilon_{t+1}.$$

The rotation R_2 generates

$$\delta = \beta_2 - b \begin{pmatrix} e_1^{*\prime}\beta_2' b & \cdots & e_{r-1}^{*\prime}\beta_2' b \end{pmatrix} - c \begin{pmatrix} e_1^{*\prime}\beta_2' c & \cdots & e_{r-1}^{*\prime}\beta_2' c \end{pmatrix},$$

which guarantees $b'\delta = c'\delta = 0$, implying that $sp(\delta) \subset sp(g_\perp)$ for $g = (b, c)$. It also follows that

$$\alpha^\circ = (a + \alpha\beta' a, \alpha)\, R_1 R_2 \quad \text{and} \quad \mu^\circ = R_2^{-1} R_1^{-1} \begin{pmatrix} \kappa^* \\ \mu \end{pmatrix}.$$

■

B Data definitions and sources

Details of the definitions of the data analysed in Section 4 and their sources are provided below.

B.1 Data definitions

π_t	=	the annual (year-on-year) rate of inflation calculated from the Consumer Price Index (CPI), expressed as a decimal.
π_t^e	=	the annual rate of expected CPI inflation (1 year out) based on surveys of expectations, expressed as a decimal.
y_t	=	the log of the production-based real Gross Domestic Product, seasonally adjusted.
i_t	=	the overnight interbank cash rate, quarterly average of monthly data, expressed as a decimal.

B.2 Sources

All the data were obtained from the website of the Reserve Bank of New Zealand (accessed on 14 June 2024). Detailed sources are as follows:

π_t	-	https://www.rbnz.govt.nz/statistics/series/economic-indicators/prices
π_t^e	-	https://www.rbnz.govt.nz/statistics/series/economic-indicators/survey-of-expectations
y_t	-	https://www.rbnz.govt.nz/statistics/series/economic-indicators/gross-domestic-product
i_t	-	https://www.rbnz.govt.nz/statistics/series/exchange-and-interest-rates /wholesale-interest-rates

C DGPs in the Monte Carlo simulation

The baseline data-generating process for the study in Section 2.3 is commonly formulated as a CVAR(1) process:

$$\Delta X_t = \alpha \left(\beta' X_{t-1} + \mu\right) + \varepsilon_t,$$

where ε_t is a multivariate *i.i.d.* pseudo normal process, $N(0, d^2\Omega)$. Here, Ω is a positive definite symmetric matrix, each diagonal and off-diagonal element assigned a unit value and a quarter, respectively, along with a damping factor $d = 0.01$. The parameters for the above process as well as a set of selection vectors a and b vary according to p and r as follows:

p	r	α	β	μ
3	1	$(-0.2, 0.1, 0)'$	$(1, -1, 1)'$	-0.01
3	2	$\begin{pmatrix} -0.2 & 0.1 & 0 \\ 0 & -0.1 & 0 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}'$	$(-0.01, -0.13)'$
4	1	$(-0.2, 0.1, 0, 0)'$	$(1, -1, -0.5, 1)'$	0.015
4	2	$\begin{pmatrix} -0.2 & 0.1 & 0 & 0 \\ 0 & -0.1 & -0.2 & 0 \end{pmatrix}'$	$\begin{pmatrix} 1 & -1 & -0.5 & 1 \\ 0 & 1 & 1 & -0.5 \end{pmatrix}'$	$(0.015, -0.08)'$
4	3	$\begin{pmatrix} -0.2 & 0.1 & 0 & 0 \\ 0 & -0.1 & -0.2 & 0 \\ 0 & 0 & -0.1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & -0.5 & 1 \\ 0 & 1 & 1 & -0.5 \\ 0 & 0 & 1 & -2.0 \end{pmatrix}'$	$(0.015, -0.08, 0.03)'$

p	r	a	b
3	1, 2	$(0, 0, 1)'$	$(1, 0, 0)'$
4	1, 2, 3	$(0, 0, 0, 1)'$	$(1, 0, 0, 0)'$

The parameters above are selected on the basis of a typical empirical study involving inflation rates and short-term interest rates, along with other macroeconomic series. The initial values X_0 range from 0.02 to 0.05, mimicking plausible inflation and interest rates. In each replication of the Monte Carlo study, 30 initial observations are discarded to mitigate the impact of the initial values.

D Further results from the Monte Carlo simulation

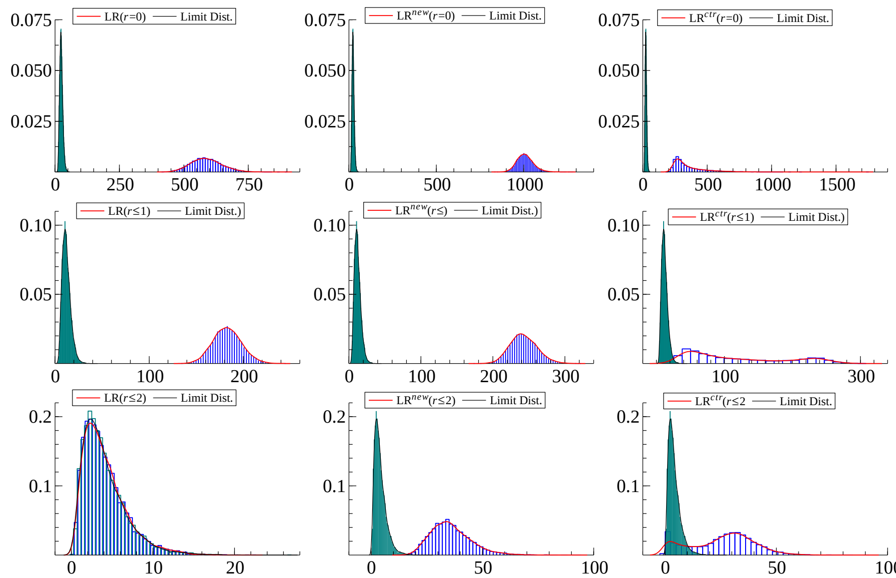

Figure 8: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 3$ and $r = 2$.

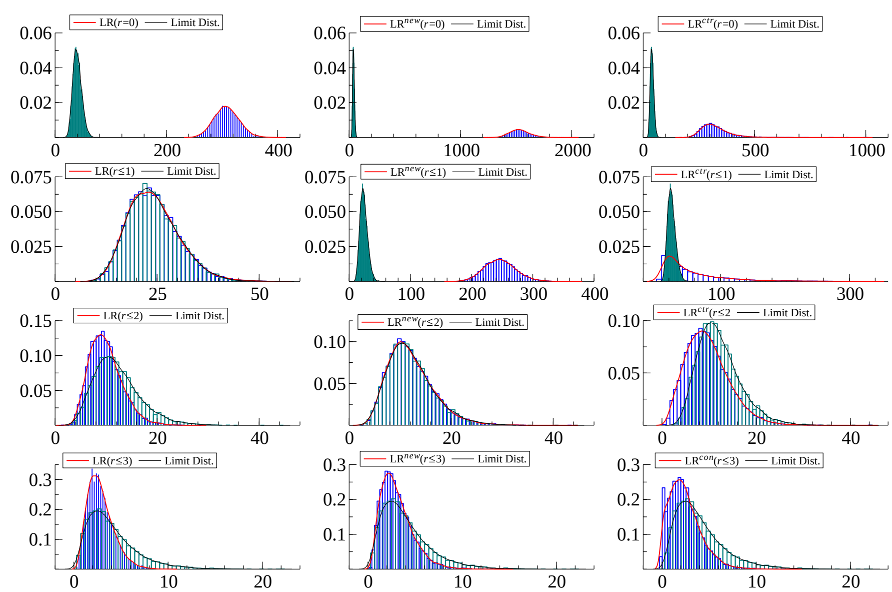

Figure 9: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 4$ and $r = 1$.

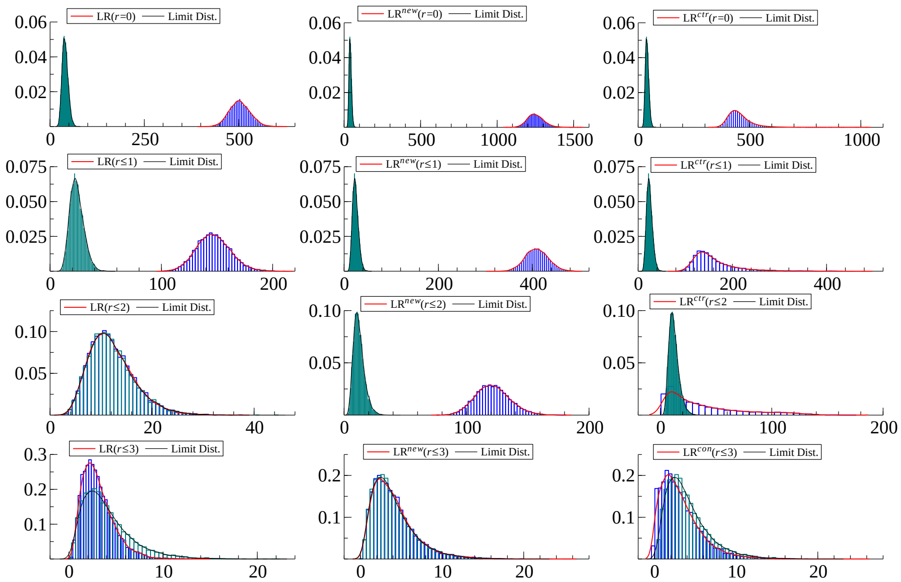

Figure 10: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 4$ and $r = 2$.

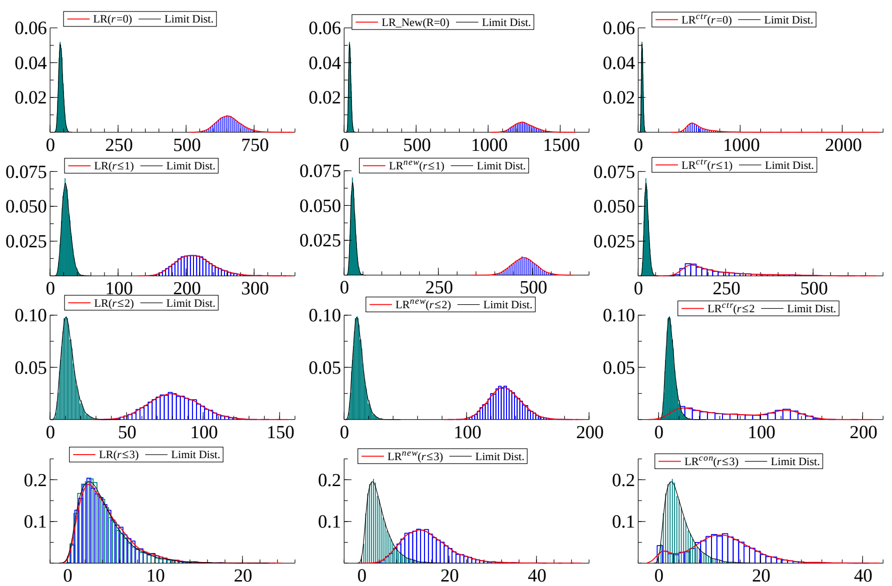

Figure 11: Distribution of cointegration test statistics at a sample of $T = 1,000$ observations for $p = 4$ and $r = 3$.

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