Are Money Demand Equations Still Alive and Kicking? Historical Evidence of Cointegration for the UK, Using Nonlinear Techniques

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Abstract:

Since the influential works of Friedman and Schwartz (1963, 1982) and Hendry and Ericsson (1991), on the monetary history of the United States of America and the United Kingdom from 1876 to 1975, there has been a great concern in the literature about the instability of money demand functions. This concern together with the results of the New Keynesian´s models (Woodford, 2003), produced the abandon of money as an instrument of monetary policy. Recently, using M1 as the measure of money, Benati, Lucas, Nicolini and Weber (2021) have shown, for a shorter and recent period of time, that there is a stable long-run money demand for a long list of countries. However, to date there are no studies showing that stable long-run and short-run money demand equations exist since the XIX century and how it can be used to inform monetary policy based on the quantitative theory of money. By means of nonlinear cointegration and nonlinear error-correction models, this paper presents evidence of UK stable long-run and short-run money demands of real broad monetary balances from 1874 to 2023. These equations provide with key elements to identify periods of excess money demand generating periods of 6.5% excess inflation, over the historical 2.2% average. Stable Money demand estimates provide useful policy rules and additional cross-check instruments for monetary policy to reach inflation targets. Furthermore, they help identifying spurious transmission channels of monetary policy, when theoretical models impose invalid common factor restrictions.

Keywords: Money Demand Stability; Nonlinear Cointegration; Nonlinear Equilibrium Correction; Nonlinear Error Correction; Opportunity Cost of Holding Money; Role of Money in Monetary Policy.
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1. INTRODUCTION

Since the seminal research by Friedman and Schwartz (1963, 1982) on the monetary history of the US and the UK from 1876 to 1975, there has been a great concern in the literature about the potential instability of money demand functions. According to the traditional prescription of the Quantity Theory of Money (QTM) expressed in growth rates ($\Delta p_t \equiv \Delta m_t - \Delta y_t + \Delta v_t,$), if the velocity of circulation of money is close to being constant ($\Delta v_t \approx 0$), central banks could achieve a zero inflation rate ($\Delta p_t = 0$) by setting the growth rate of money supply equal to the growth rate of real income ($\Delta m_t = \Delta y_t$). As a result, the stability of money demand equations became a central tenet in the design of conventional monetary policy from early 1970s until the mid-1990s, developing into one of the best researched fields in applied macroeconomics. Over several decades, literally hundreds of papers have been published that contain empirical money demand estimations for many countries and time periods with quite diverse findings. The range of the estimated income and interest-rate elasticities is wide, and while some papers maintain that money demand is stable, others reach the opposite conclusion.

Among those defending stability, David Hendry´s empirical contributions to this literature have been extremely influential, especially regarding the UK money demand over long historical periods ending in the early 1990s. As illustrated in Hendry and Ericsson (1991)´s revaluation of Friedman and Schwartz (1982)´phase-average results for the UK, the use of error-correction (cointegration) techniques were able to restore the short-run and long-run stability of this function. However, despite these favorable results, new evidence about unpredictable shifts in money velocity, particularly in the US, implied the withdrawal of monetary aggregates as the main tool used by central banks to control monetary policy. For example, the US Fed already de-emphasized their role in the 1990s, while the German Bundesbank faced severe difficulties with meeting short-term monetary growth during that decade. Faced with these problems, the new monetary paradigm switched from money growth intermediate targets to the choice of inflation (and output) targets monitored through policy rules (i.e. Taylor rules) steering short-term interest rates.

This tendency was reinforced by the rise of a class of New Keynesian (NK) models that were able to explain fluctuations in key macroeconomic time series abstracting from shifts in money velocity or allowing them to mirror movements in real balances. The NK model was considered as the right framework to think about optimal monetary policy in economies subject to nominal rigidities, leading to trade-offs between inflation and the output gap in the short run. Money neutrality only held in the long run while monetary aggregates played no direct role whatsoever in the transmission of monetary policy to output and inflation at higher frequencies. Monetary policy decisions were made regarding the targeted nominal interest rate (the Bank Rate in the case of the Bank of England) whose changes, due to price rigidity, influenced the real interest rate. The latter influences aggregate demand via the gap between actual output and the economy's potential output that would be realized if prices were fully flexible. Changes in the output gap in turn impact on inflation via the NK Phillips curve. Of course, in this framework, the supply of money is influenced by the open-market operations that the central bank conducts to achieve the intended rate of interest, so that that actual money growth then results from the interplay of money supply and money demand in a recursive fashion. In other words, central banks supply sufficient money to satisfy demand for real balances at the intended rate of interest, the current price level and current income. Consequently, the optimal interest rate policy was characterized without any recourse to monetary aggregates. From this perspective, efforts to improve measures of the money supply or to obtain better empirical estimates of the parameters ruling money demand were assessed as being pointless to improve the performance of monetary policy (Woodford 2003). Faced with these visions there was however an alternative branch of the literature which defended the stability of money demand at low frequencies to determine the price level, particularly in situations where the output gap or the real interest rates were subject to measurement errors, calling for a two-pillar monetary strategy, like the one implemented by the ECB. According, to this strategy, Taylor rules would be implemented in normal times whereas money growth target would play a role to discipline inflation in times of turmoil.

Yet, with the arrival of the Great Recession in 2008, the simple decision rules in operation for two decades partly became ineffective in a near-zero interest rate regime. They were replaced by unconventional monetary policy tools, such as large-scale asset purchase (LSAP) programs or quantitative easing (QE). At the zero-lower bound, being unable to achieve any further reductions in the short-term nominal interest rate that banks charge each other for an overnight loan, central banks implemented large-scale purchases of longer-term maturities, including government and corporate bonds and even stocks. As a result of these actions, the amount of money circulating in several economies increased substantially, aiming to boost aggregate demand through cuts in the cost of borrowing by households and firms. Yet, for that strategy to succeed, money demand ought to remain stable. Hence, with the increase of uncertainties regarding monetary policy further accentuated during the onset of the pandemic crisis in 2020, understanding the dynamics of the demand for real money balances has experienced a revival, emerging again as a relevant issue in current discussions about monetary policy.

In view of these circumstances, this paper revisits the money demand specifications proposed by David Hendry and co-authors in the past spanning more than a century of data (1878-1993 in his latest contribution to this topic; cf. Ericsson, Hendry, and Prestwich (1998)), and others UK money demand competing models by Escribano (1985, 1986, 2004) and Teräsvirta and Eliasson (2001). We address to what extent its main features remain valid in modelling long-run trends and short-run variations in money velocity using an updated sample which ends in 2023 and therefore covers the Great Recession and the beginning of the Great Contagion. In doing so, we aim at improving our ability to interpret the influence of broad money growth on future nominal GDP growth in a way that accords with basic monetary theory. Our main findings are the money demand specifications proposed two decades ago remain valid even after major events, like the Great Recession, the banking crisis and the covid-pandemic. One key finding is that both money growth and the pace of recovery in velocity can help track changes in real income and in interest rates growth needed to return inflation to its target.

We show in this paper that the main five robust empirical UK drivers of nominal money since the XIX century are: i) In the long run, the evolution of prices, real income (both with an elasticity equal to one) and short-term nominal interest rate (cointegration) based on the Quantity Theory of Money. ii) In the short run, or with high frequency fluctuations, the main drivers of the evolution of the rate of growth of real balances are the rates of growth of prices (inflation), rate of growth of short-term interest rates, the rate of growth of long-term interest rates and the equilibrium corrections terms towards the previous long-run equilibria (nonlinear equilibrium-correction). Furthermore, iii) real income is an important determinant in long-run money demand but its rate of growth plays no direct (only indirect) role on the rate of growth of real money demand, once we eliminate this spurious relation in rates of growth obtained by imposing a common factor restriction. iv) The coefficients of the money demand variables in levels (cointegration) and in rates of growth are different (no common factor restriction accepted), mainly because the variables of the equation in differences, real income growth and changes in short term interest rates, are not exogenous, with important monetary policy implications. And finally, v) periods of excess real money demand, due to exogenous shocks like wars, regulatory changes and COVID, are responsible of an additional 6.5% increase in unconditional inflation, over the historical average of 2.2% inflation´s rate. Therefore, the information obtained estimating stable long-run and short-run real money demands, provides additional cross-check instruments to anticipate future inflationary periods, caused by periods of excess money demand (excess liquidity).

In this context, the paper is structured as follows: Section 2 reviews the main empirical literature on money demand functions; Section 3 introduces and describes the historical data for the UK; Section 4 analyzes nonlinear equilibrium correction models (NEC) and for tests for cointegration based on Monte Carlo Simulations; Section 5 presents and estimates the long-run cointegration equilibrium and the NEC empirical models of UK money demand; Section 6 assesses the monetary policy implications of our empirical findings; and Section 7 concludes.

2. WHY MONEY DEMAND?

In recent decades, economists and central banks have shifted their focus from monetary aggregates to interest rate rules and inflation stabilization policies. What was once considered the cornerstone of monetary policy until the mid-1980s has taken a backseat in modern macroeconomic theory and practice, as seen in New-Keynesian Models (Woodford 2003; Galí 2007). This transition, however, has not been without empirical justification.

Historically, shocks to real money demand have been volatile and persistent, making money targeting prone to introducing substantial policy outcome volatility (Canova and Menz 2011; Benati et al. 2021). Interest rate rules, on the other hand, have demonstrated immunity to such shocks and proven successful in stabilizing inflation rates. Traditional models have struggled to maintain stability over extended periods, particularly with post-1970s data, leading to implausible parameter estimates, autocorrelated errors, and poor forecasts (B. M. Friedman and Kuttner 1992). Further, the instability of monetary aggregates observed in the late 1990s and the weakening relationship between these aggregates and real economic activity in the short run cast doubt on money demand as a reliable policy tool.

The financial crisis reignited discussions on optimal monetary policy implementation, particularly when interest rates approached the effective lower bound (ELB). Under such constraints, money-based rules may outperform traditional frameworks, as suggested by Belongia and Ireland (2019). Additionally, some evidence points to the value of incorporating money into interest rate rules like the Taylor´s Rule, enhancing model accuracy and effectiveness in inflation control (Qureshi 2021). Nonetheless, the role of money as a dependable economic indicator hinge on the ability to forecast and model monetary aggregate trends consistently. In this regard, the findings of Benati et al. (2021), which highlight the stability of M1 long-run demand across 38 countries over extended sample periods, are particularly significant. Yet, concerns about stability persist, both in the short and long run, especially for broader monetary aggregates.

In this context, this paper examines the main drivers of nominal and real UK money demand since the 19th century, uncovering four empirical regularities that are robust under alternative and competing dynamic nonlinear specifications.

2.1. Long-run Money Demand and QTM

Understanding of money demand relationships is crucial for reducing the risk of implementing destabilizing monetary policies (Ball 2012). Many of the core principles underlying this understanding were established over half a century ago by monetarist theories, which are deeply rooted in the Quantity Theory of Money (Fisher 1911; M. Friedman 1956). These theories emphasize the significant role of money in driving economic fluctuations. According to monetarists, if the velocity of money—or its inverse, the demand for money—can be modeled as a stable function of the nominal short-run interest rate (M. Friedman 1956; 1961; Selden 1956; Laidler 1993) then the aggregate price level, real income, and interest rates emerge as the primary long-run determinants of money demand, consistent with the quantity of money.

Since its introduction by Fisher (1911), the classical QTM has formed the basis of many money demand theories. These models often incorporate the transactions demand approach, which was further developed through the well-known square root formula linking money and interest rates. This formulation draws from the seminal works of

Baumol (1952) and Tobin (1965), which provide a foundational framework for understanding the dynamics between money demand and interest rates in both theoretical and practical contexts.

A simple theoretical derivation of the long-run money demand based on QTM is the following. Let, $MV = PY$ where V= velocity of circulation of money, M= nominal money stock, P= output deflator, and Y= real output. In equilibrium, $M^d = M^s = M$. According to QTM, P is the endogenous variable. In the money demand approach, M is the endogenous variable and $\frac{M}{PY} = \frac{1}{V} \Leftrightarrow M = \frac{1}{V}PY$.

Assuming that velocity is $V = f(RS)$, where RS is the nominal short-run interest rate, with $f'(RS) > 0$, so that $1/_V = g(RS)$, with $g'(RS) < 0$. Note[1] that RS could also be formulated as the spread between the alternative (bond) and own (deposit) interest rates. Taking logs (small letters) yields $m - p - y = \log g(RS)$. In log-linear form becomes, $lng(RS) = \alpha - \delta$rs or in semi-log form $m - p - y = \alpha - \delta$RS.

The well-known Baumol-Tobin money demand equation, Baumol (1952) and Tobin (1965), can be derived as follows. Suppose I get nominal income PY which is deposited in a bank account which is fully withdrawn in one period, whose length is normalized to 1. Since at the beginning of each period there is an amount PY and at the end there is 0, the average amount of money I withdraw each period is $PY/_2$. Similarly, if I initially withdraw half of my income, spend it, then in the middle of the period go back to the bank and withdraw the rest, I have made two withdrawals ($n = 2$) and my average money holdings M are equal to $M = PY/_4$. In general, the person's average money holdings will be $M = PY/_{2n}$. Assume that the nominal cost of these transactions is $c(n) = \psi\, n^\sigma$ with $\sigma > 0$. Likewise, the amount of interest income lost per period due the withdrawals is $RS * Y/_{2n}$. Hence[2], n is chosen to minimize total costs of transactions and foregone interest income, that is,

$$n^* = \arg\min\ \psi\, n^\sigma + \frac{RS * PY}{2n}.$$

[1] We could also introduce in $g(.)$, dummy variables to capture changes in financial regulation, wars, COVID periods, etc. as we obtain in the empirical application.

[2] We have benefited from discussions with Juan J. Dolado about the unreliable role of money for inflation control. Furthermore, he suggested us this alternative theoretical derivation of the long-run money demand.

whose first order condition is,

$$\psi\sigma n^{\sigma-1} = \frac{RS * PY}{2n^2} \Leftrightarrow n^* = \left(\frac{RS * PY}{2\psi\sigma}\right)^{\frac{1}{1+\sigma}}$$

Money demand per period is then given by

$$M^* = \frac{PY}{2n^*} \approx= PY^{\frac{\sigma}{1+\sigma}} \Big/ RS^{\frac{1}{1+\sigma}}.$$

Dividing by P and taking logs yields the log-log equation,

$$m - p = \sigma/(1+\sigma)\, y - 1/(1+\sigma)\ \text{rs},$$

such that for $\sigma = 1,$ yields Baumol-Tobin´s square root specification with income and interest rate elasticities given by 0.5 and -0.5, respectively. A problem with this specification of the interest rate is that changes in logs are akin to percentage points, which are the units of RS. Thus, rs does not have meaningful units and a 0.5 increase/reduction in short-term interest rates have the same effect no matter whether the level of the interest rate is high or low. This is solved by expressing the long-run money demand in semi-log form, $\ln g(RS) = \beta_0 - \beta_{RS} RS$ leading to,

$$m - p - y = \beta_0 - \beta_{RS} RS$$

where now changes in all variables are measured in the same units.

Woodford (2003), Ireland (2004), Ansdres et al. (2006), Benati et al. (2021), proposed alternative stochastic approaches to derive relations among money, income, prices, and nominal short-run interest rate, able to explain the long-run behavior (at low spectrum frequencies) of money demand. New-Keynesian modeling approaches, see Woodford (2003), of monetary policy suggest that the short-term nominal interest rate is the major policy instrument of central banks to control inflation. The optimal level is based on inflation forecasts and output gaps (demand shocks and cost-push shocks), independent on monetary aggregates (money demand parameters do not enter here) but none of them are observable variables.

Lucas (2007) mentioned that this New-Keynesian´s models cannot explain the inflation of the 1970´s where the cyclical trends in money growth and inflation moves in a parallel way. This empirical regularity is consistent with the QTM. Lucas (2007) supports the inclusion of monetary aggregates to cross-check on interest rate policies.

formulations, based the European Central Bank (ECB) *two pillars policy* (one based on setting nominal short-term interest for controlling inflations and two based on cross-checks based on the QTM linking money growth and inflation). Wieland et al. (2010) show that including persistent central bank misperceptions into New-Keynesian´s models can generate similar cyclical trends in inflation and growth in monetary aggregates, mentioned before and included in the graphs of Benati (2005). Modern Keynesian´s monetary policy does not consider money as a useful instrument to control inflation (Galí 2007; Woodford 2007), not even in the presence of a stable money demand. Galí (2007) says, "*The value of monitoring monetary aggregates or measures of excess liquidity as part of an assessment of the risks to price stability is questionable, even in the presence of a stable money demand*." However, we argue in this paper that we get additional and useful monetary policy information from having stable money demand estimates.

In the following sections, selected UK empirical models in the literature will be replicated first using the original samples periods, then with a mechanically extended to 2023 to identify main changes and finally new specifications for each formulation will be suggested. We will also discuss other key empirical issues like, alternative broad money measures, alternative measures of the opportunity cost of holding money, alternative functional forms of long-run money demand (nonlinear cointegration) and alternative short-run functional forms of money demand (nonlinear equilibrium correction, smooth transition models).

3. HISTORICAL DATA FOR THE UK (1871 - 2023)

We now present and describe the historical data and its properties to determine the class of econometric models to consider in the empirical modeling of the money demand in the long run (cointegration), and in the short run (dynamic models).

The basic data encompasses annual figures of the nominal broad money (M_t), nominal high-powered money (H_t), real income (Y_t), short-run (RS_t) and long-run (RL_t), interest rates and prices (P_t). The sample period spans the period from 1871 until 2023. Data have been mainly retrieved from Escribano (2004) and Thomas and Dimsdale (2017) and updated appropriately (see Data Appendix A). This dataset for the UK was originally constructed by Friedman and Schwartz (1982) and later rescaled by Hendry and Ericsson (1991) to compensate for the Southern Ireland (Republic of Ireland) effect.

Uppercase letters are used for variables in levels, while *lowercase* letters denote variables in natural logarithms and delta (Δ) indicates the difference operator[3]. Appendix A elaborates in more detail the data sources and the definitions of the variables used. Based on these data, we have constructed the natural logarithms of real money balances $(m-p)_t$, the natural logarithm of real income (y_t), the opportunity cost measure of holding money in levels (RNA_t) and in logs with lower letters (rna_t) and the inflation rate (Δp_t)[4]. These variables integrate four key historical monetary trends derived from the Quantity Theory of Money (see Figure 1): the link between short-run interest rates, real income, real money balances, and the response of prices to money growth. These relationships, previously documented in the UK by Friedman and Schwartz (1982), Hendry and Ericsson (1991), among others, provide the foundation for constructing empirical money demand and inflation equilibrium-correction models study afterwards.

As proposed by Ericsson, Hendry, and Prestwich (1998), we keep the use of the opportunity cost measure $RNA_t \equiv RS_t * \left({H_t}^A/{M_t}^A\right)/c$, where RS_t is the short-term interest rate, ${H_t}^A/{M_t}^A$ is the proportion of high-powered money to broad money using actual values (not spliced), and c is a constant rescaling factor equal to 0.25. Unlike traditional interest rate measures, this measure is particularly effective in capturing substitution processes among historical monetary components, as it adjusts short-term interest rates (RS_t) to reflect changes in the composition of money (${H_t}^A/{M_t}^A$), especially during periods of definitional changes in monetary aggregates (see Figure B.1). By relying on actual values of monetary components without splicing, this measure ensures a more accurate representation of real movements, making it highly appropriate for analyzing broad money dynamics and ensuring the robustness of empirical results.

The unit root (and other possible sources of no stationarity) analysis in the data corroborates what is usually reported in the literature: real money balances, real income, prices, and interest rates are I(1) processes. Standard unit-root analysis such as the Augmented Dickey-Fuller (ADF) test, the correlograms, and the plotted time series, as

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[3] The difference operator Δ is defined generally as $\Delta_j^i x_t = \left(1-L^j\right)^i x_t$ for $i > 0$ and $j > 0$, where L is the lag operator that shift the variable *j* periods into the past, and *i* is the number of differences taken. If *i* or *j* are not explicit, it is assumed to be unity (i.e., $\Delta x_t = x_t - x_{t-1}$). For convenience in some graphs this operator is presented as "*d*" followed by the variable name at period *t* in in parenthesis (i.e., *d_mp (-1)* = $\Delta(m-p)_{t-1}$).

[4] The detailed discussion on descriptive stats, variable selection, limitations, and sensitivity analysis for each specific variable used in this empirical application can be found in-depth in Appendix B of Escribano and Rodriguez (2023, p. 127-51).

well as nonparametric tests (Rank Unit Root Test) of Escribano, Sipols, and Aparicio (2006a) robust to monotonic nonlinear transformations of the variables can be found in Appendix B of Escribano and Rodriguez (2023, p. 127-51).

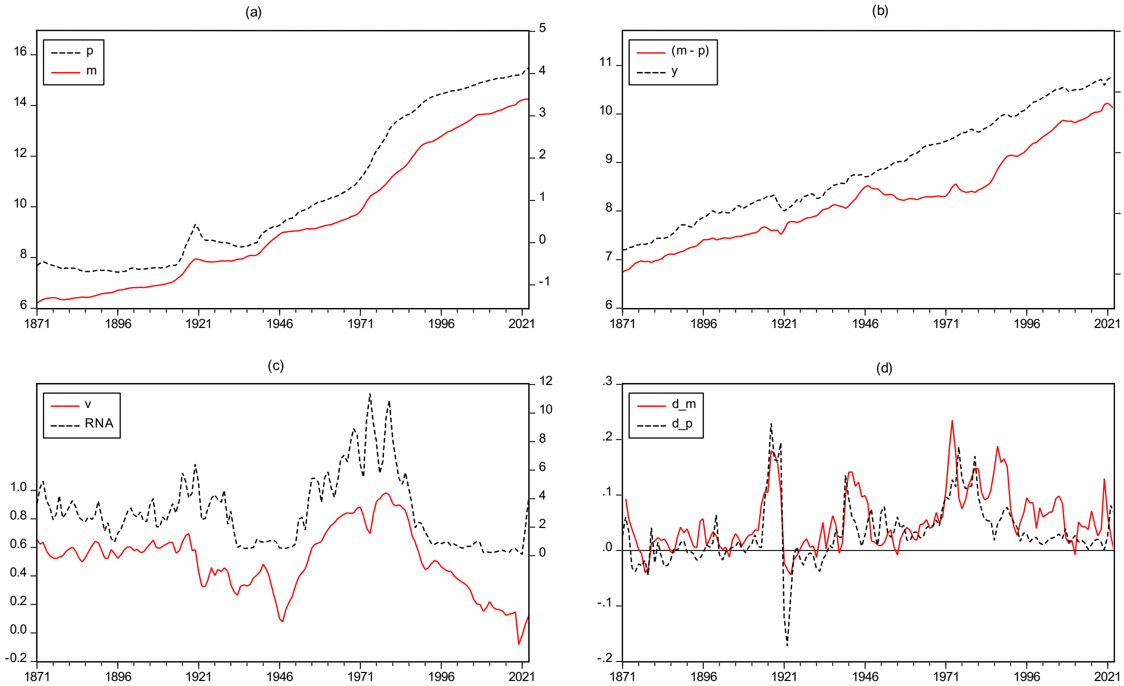

Figure 1. Main Historical Monetary Trends (1871 – 2023)

Notes: (a) nominal broad money and prices in logs (m_t and p_t), (b) real money balances and real output in logs ($(m-p)_t$ and y_t), (c) money velocity in logs and adjusted short-run interest rates in percent per annum (v_t and RNA_t), and (d) nominal money`s growth and the inflation rate (Δm_t and Δp_t). Note that in (a), (b) and (c) the left axis corresponds to the series with a dashed line. Source: see data Appendix A.

For the modelling of money demand, we focus on broad money measures due to its higher degree of "moneyness" and its ability to capture substitution processes among various monetary assets, offering a more accurate picture of monetary trends and growth. While narrow money, consisting of highly liquid assets, remains valuable for studying high-liquidity movements and cross-country comparisons — as shown recently by Benati et al. (2021) — broad money measures provide a more stable basis for analyzing long-term relationships and monetary dynamics. Broad money's stability and its alignment with economic indicators like output, and inflation make it particularly suitable for this analysis, especially given the availability of long historical series in the UK dating back to the late 19th century.

4. NONLINEAR EQUILIBRIUM CORRECTION MODELS (NEC) AND COINTEGRATION TESTING: MONTE CARLO SIMULATIONS

Consider the following bivariate, y_t, x_t, cointegrating system (1.1) - (1.3), with $-1 < \rho_1 < 1$, η_t and ε_t , i.i.d. N(0,1). Those conditions are sufficient for z_t to be I(0), x_t I(1) and from (1.1) y_t with $\beta_x \neq 0$ is also I(1).

$$y_t = \beta_0 + \beta_x x_t + z_t \tag{1.1}$$

$$z_t = \rho_1 z_{t-1} + \eta_t \tag{1.2}$$

$$\Delta x_t = \varepsilon_{x,t} \tag{1.3}$$

An equivalent and useful reparameterization of (1.2) is the following Dickey-Fuller (1979) testing equation, where now $-2 < (\rho_1 - 1) < 0$,

$$\Delta z_t = (\rho_1 - 1) z_{t-1} + \eta_t$$

A simple and useful generalization to nonlinear error correction (NEC) models is the following,

$$y_t = \beta_0 + \beta_x x_t + z_t \tag{2.1}$$

$$\Delta z_t = \beta_1 z_{t-1} + \beta_2 z^2{}_{t-1} + \beta_3 z^3{}_{t-1} + \eta_t \tag{2.2}$$

$$\Delta x_t = \varepsilon_{x,t} \,. \tag{2.3}$$

A usual condition for z_t to be I(0) is that the nonlinear function on the right of (2.2) is asymptotically bounded by a linear function of z_t (for large absolute values of z_t the nonlinear function is dominated by a linear function), Saikkonen (2005), Kapetanios, Shin, and Snell (2006), and Kiliç (2011). For that, if there is parameter multiplying a nondecreasing nonlinear function, this parameter must be negative, so that the nonlinear function is decreasing. And the opposite if the nonlinear function is decreasing. The condition now for z_t to be I(0) is that the first derivative of (2.2) with respect to z_{t-1} should be negative, as will be seen by simulations below. Sufficient conditions for this NEC-test are that β_1 and β_3 are negative with $(\beta_1 + 3\beta_3 z^2{}_t) < -2\beta_2 z_t$. Testing in (2.2) the null hypothesis (H_0) that the first derivative of cubic polynomial is zero against the alternative hypothesis (H_1) that is negative, is a nonlinear version of the well-known EG-test.

To obtain from (2.1) - (2.3) a nonlinear error-correction (equilibrium correction) model without imposing the COMFAC restriction, we set $\eta_t = \beta_z \Delta x_t + \varepsilon_{z,t}$, and taking first differences in (2.1) together with (2.2) to get (3.1), where $\delta = \beta_x + \beta_z$,

$$\Delta y_t = \beta_1 z_{t-1} + \beta_2 z^2_{t-1} + \beta_3 z^3_{t-1} + \delta \Delta x_t + \varepsilon_{z,t} \tag{3.1}$$

$$\Delta x_t = \varepsilon_{x,t}. \tag{3.2}$$

Testing now in (3.1) the null hypothesis (H_0) that the first derivative of the cubic polynomial is zero against the alternative hypothesis (H_1) that is negative, is a nonlinear version of the error correction test, NEC-test.

A Simple Bivariate Nonlinear Granger´s Representation Theorem is,

(i) If the variables y_t and x_t from equations (2.1) – (2.3) satisfy the condition that the first derivative of (2.2) w.r.t. z_{t-1} is negative (and asymptotically bounded by a linear function), then they are cointegrated and have a nonlinear error correction model, given by (3.1) – (3.2).

(ii) If the nonlinear error correction given by equations (3.1) – (3.2) satisfy that the first derivative of (3.1) w.r.t z_{t-1} is negative (and asymptotically bounded by a linear function), then variables y_t and x_t are cointegrated.

In general, if the variables are cointegrated, we know that there is at least an equilibrium-correction model in one of the two single ARDL(p,q) models. But we do not know if the equilibrium correction is linear or nonlinear. To get intuition on the previous cointegrated condition in small samples, that the first derivative of the nonlinear error correction term with respect to z_{t-1} must be negative, we generate data in Figure 2. The data generating process (DGP) is the following,

$$y_t = 2 + x_t + z_t \tag{4.1}$$

$$\Delta z_t = f(z_{t-1}) + \eta_t \tag{4.2}$$

$$\Delta x_t = \varepsilon_{x,t} \tag{4.3}$$

where η_t and $\varepsilon_{x,t}$ are i.i.d. N(0,1). The cubic polynomial NEC is

$$f(z_{t-1}) = \beta_1 z_{t-1} + \beta_2 z^2_{t-1} + \beta_3 z^3_{t-1}$$

In Figure 2, we represent the system (4.1) to (4.3) with parameter values equal to β_1= -0.005, β_2= 0.003, β_3= -0.00092, for the decreasing polynomial (error correcting). In the first row of each block of graphs, we plot the two series y_t and x_t through time, and clearly they are nonstationary and I(1). The decreasing cubic-polynomial error correction is in the graph to the right of it. In the second row, we plot the residuals of the cointegrating equation and they looks I(0), or stationary as expected. Finally, the second graph of the second row shows that the first derivative of the NEC is negative (only equal to 0 at equilibrium) for all values of z_t. Similar graphs, but for the Logistic are in Escribano et al. (2025).

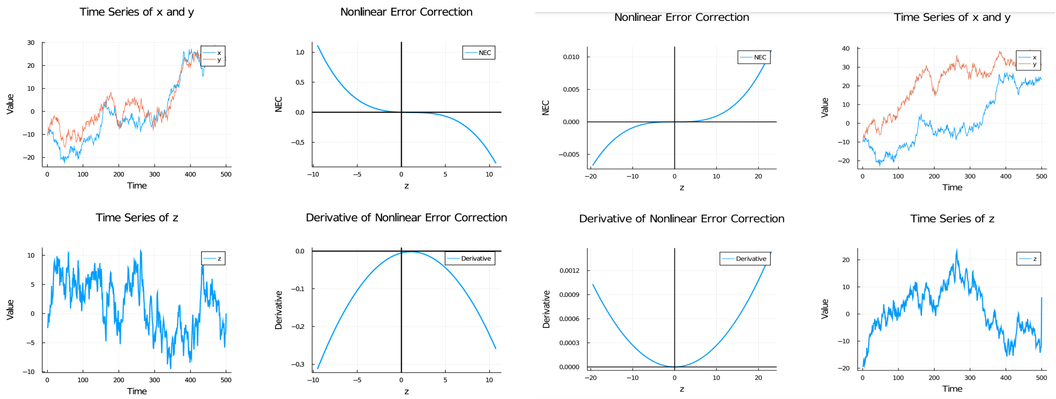

Figure 2. *Decreasing (Left)* and *Increasing (Right) Cubic Polynomial Error-Correction*

Notes: This figure illustrates the system defined by equations (4.1) to (4.3), using parameter values of β_1 = -0.005, β_2 = 0.003, and β_3 = -0.00092 for the decreasing cubic-polynomial error correction and β_1= 0.005, β_2= −0.003, β_3= 0.00092 for the increasing case shown on the right-hand side. Source: Authors' calculations.

In Figure 2, we also represent the system (4.1) to (4.3) for the following parameter values equal to β_1= 0.005, β_2= −0.003, β_3= 0.00092, for increasing polynomials (non-error correcting) that do not generate stationary equilibrium errors (z_t).

4.1. Monte Carlo Simulations

Monte Carlo Simulations are based on the data generating process (DGP) of Arranz and Escribano (2000) that was also used by Kapetanios et al. (2006). Under the null hypothesis (H_0) that variables y_t and x_t are I(1) and not cointegrated, the data generating process (DGP) is,

$$\Delta y_t = \delta \Delta x_t + \varepsilon_{z,t} \tag{5.1}$$

$$\Delta x_t = \varepsilon_{x,t} \tag{5.2}$$

with η_t and ε_t , i.i.d. N(0,1), δ = 0, 0.5, 1, x_0 = 0 and y_0 = 0, the sample size T=150, 500. Burn-in B=50, 50 initial observations are discarded, M=2000 is the number of replications, K.

Under the alternative hypothesis (H_1), that variables y_t and x_t are I(1) and cointegrated with a NEC-cubic polynomial model[5], the data generating process (DGP) is,

$$y_t = \beta_0 + \beta_x x_t + z_t \tag{6.1}$$

$$\Delta y_t = \beta_1 z_{t-1} + \beta_2 z^2_{t-1} + \beta_3 z^3_{t-1} + \delta \Delta x_t + \varepsilon_{z,t} \tag{6.2}$$

$$\Delta x_t = \varepsilon_{x,t} \tag{6.3}$$

The first derivative of the cubic polynomial is negative if $(\beta_1+2\beta_2 z_t +3\beta_3 z^2_t)<0$. Empirically, we can always plot the first derivate for all values of z_t to check if is always negative. But in order to propose a simple and useful EC-test we can test whether β_1 and/or β_3 are, jointly or individually, significant and negative in either the NEC-test (6.2) or the nonlinear version of Engle and Granger (1987) test, NEG-test, based on equation (7). In (7) we are imposing the COMFAC restriction that δ in (5.1) is equal to β in (1.1). To get the nonlinear version of the Dickey-Fuller (1979) test, or EG-test, based on the cointegrating errors z_t of (6.1), we have,

$$\Delta z_t = \beta_1 z_{t-1} + \beta_2 z^2_{t-1} + \beta_3 z^3_{t-1} + \varepsilon_{z,t} \tag{7}$$

To implement this generalization of the two-step approach of Engle and Granger (1987), EG-test, we first estimate the cointegrating equation (8.1) by OLS,

$$y_t = \hat{\beta}_0 + \hat{\beta}_1 x_t + \hat{z}_t \tag{8.1}$$

$$\Delta \hat{z}_t = \beta_1 \hat{z}_{t-1} + \beta_2 \hat{z}^2_{t-1} + \beta_3 \hat{z}^3_{t-1} + \eta_t \tag{8.2}$$

Now, based on alternative parameterization of equations (8.2), we test the null hypothesis (H_0) of no- cointegration (a unit root in z_t) that β_1=0, β_2=0 and β_3=0 in three different specifications of (10.1), (10.2) and (10.3), with the augmented version of the equations to control for the possible dynamic misspecifications to obtain residuals that

[5] Notice that a cubic polynomial is only a general small sample approximation but asymptotically is unbounded and therefore it does not satisfy the condition that is bounded by a linear function. If we want to impose the asymptotically bounded condition on the polynomial function, we need to consider the alternative class of rational polynomials, as was done in Escribano (2004).

are white noise. In all cases estimated under H_1, cointegration, we expect to obtain values $\beta_1<0$ and $\beta_3<0$ and at least one of them significant. Kapetanios et al. (2006) derived the asymptotic distributions of the F-test and the alternative t-tests based on the error correction coefficients of equations (9.2) and (9.3), for the exponential smooth transition regression (ESTR). Kiliç (2011) proposed a different test statistic and apply it also to the logistic case (LSTR). Both papers provide the corresponding asymptotical critical values.

Here, we analyze the small samples effects and generate the 5% and 10% critical values (c.v.) of EC-test and NEC-test, under the null hypothesis (H_o) of no cointegration, with DGP given by equations (5.1) and (5.2). Those c.v. are independent of δ and are included in Table 1 for two sample sizes T=150 (similar size to this UK money demand data) and T=500.

Table 1. Critical Values of Linear and Nonlinear Equilibrium-Correction (EC) Tests

	$\alpha = 5\%$				$\alpha = 10\%$			
T	***EC (t_1)***	***NEC (t_3)***	***NEC (t_1^*)***	***NEC (t_3^*)***	***EC (t_1)***	***NEC (t_3)***	***NEC (t_1^*)***	***NEC (t_3^*)***
150	-1.8738	-2.3841	-3.2052	-3.1274	-1.5502	-2.0471	-2.8918	-2.8256
500	-1.8241	-2.4589	-3.1893	-3.1423	-1.4481	-2.1443	-2.8645	-2.8507

Notes: The 5% critical values (c.v.) of linear (EC) and nonlinear (NEC) cointegration tests under the null hypothesis (H_o) of no cointegration with DGP given by equations (5.1) an (5.2) for simulated sample T. Source: Authors´ calculations

The EC-test and NEC-test statistics are obtained from equations (9.1) to (9.3). The t-ratio of β_1 in in the linear EC model (9.1) is called, (t^*_1), and the t-ratio of β_1 estimated in the NEC model with the three terms of the cubic polynomial, NEC(1,2,3), in (9.3) is called (t_1). The t-ratio of β_3 when including only the cubic term, NEC(3), in (9.2) is called, (t^*_3), and the t-ratio of β_3 estimated in the NEC(1,2,3) in (9.3) is called (t_3).

$$\textbf{EC-test: } \Delta y_t = \beta_1 \hat{z}_{t-1} + \delta \Delta x_t + \varepsilon_{z,t} \tag{9.1}$$

$$\textbf{NEC(3)-test: } \Delta y_t = \beta_3 \hat{z}^3_{t-1} + \delta \Delta x_t + \varepsilon_{z,t} \tag{9.2}$$

$$\textbf{NEC(1,2,3)-test: } \Delta y_t = \beta_1 \hat{z}_{t-1} + \beta_2 \hat{z}^2_{t-1} + \beta_3 \hat{z}^3_{t-1} + \delta \Delta x_t + \varepsilon_{z,t} \tag{9.3}$$

The power of the EC (t^*_1-test) is analyzed in the testing equation (9.1) using β_1 = -0.001, -0.003, -0.05 -0.1, -0.2. In equation (9.2) for the NEC (t_{3*}-test) the values of are: β_3 = -0.001, -0.003, -0.05, and for equation (9.3), the t-tests (t_1, t_2 and t_3), with parameter values β_1 = -0.001, -0.003, -0.05 -0.1, -0.2, for β_2 =-0.03 and for β_3 = -0.001, -0.003, -

0.05. The same parameter values are used in EG test and NEG tests, of the testing equations (10.1), (10.2) and (10.3). See Figure 3 to see the type of cubic polynomials that we are considering.

Nonlinear EG-test and NEG-test are estimated based on the augmented models, (10.1) to (10.3), using p-lags of the dependent variable Δz_t, to make sure that the residuals are white noise.

EG-test: $$\phi_p(L)\,\Delta\hat{z}_t = \beta_1\hat{z}_{t-1} + \varepsilon_{z,t} \qquad (10.1)$$

NEG(3)-test: $$\phi_p(L)\,\Delta\hat{z}_t = \beta_3\hat{z}^3_{t-1} + \varepsilon_{z,t} \qquad (10.2)$$

NEG(1,2,3)-test : $$\phi_p(L)\,\Delta\hat{z}_t = \beta_1 z_{t-1} + \beta_2\hat{z}^2_{t-1} + \beta_3\hat{z}^3_{t-1} + \varepsilon_{z,t} \qquad (10.3)$$

The 5% and 10% critical values (c.v.) of alternative EG-test and NEG-test**,** obtained under the null hypothesis (H_o) of no cointegration with DGP given by equations (5.1) and (5.2), are independent of δ and included in Table 2.

Table 2. Critical Values of Linear and Nonlinear Engle-Granger (EG) Tests

	$\alpha = 5\%$				$\alpha = 10\%$			
T	***EG (t_1)***	***NEG (t_3)***	***NEG (t_1^*)***	***NEG (t_3^*)***	***EG (t_1)***	***NEG (t_3)***	***NEG (t_1^*)***	***NEG (t_3^*)***
150	-1.9949	-2.4992	-3.4188	-3.4276	-1.6057	-2.1864	-3.1373	-3.0541
500	-1.89135	-2.5189	-3.4052	-3.3798	-1.4713	-2.1628	-3.0540	-3.0706

Notes: The 5% critical values (c.v.) of linear (EG) and nonlinear (NEG) cointegration tests under the null hypothesis (H_o) of no cointegration with DGP given by equations (5.1) an (5.2) for simulated sample T. Source: Authors´ calculations

The EG-test and NEG-test statistics are estimated from equations (10.1) to (10.3). The t-ratio of β_1 in the linear EG equation (10.1) is named, (t^*_1), and the t-ratio of β_1 estimated in the NEG model with the three terms of the cubic polynomial, NEC(1,2,3), in (10.3) is named (t_1). The t-ratio of β_3 when including only the cubic term, NEC(3), in (10.2) is (t^*_3), and the t-ratio of β_3 estimated in the NEC(1,2,3) in (10.3) is named (t_3).

To evaluate the power of the alternative linear and nonlinear cointegration test, alternative DGPs are generated under H_1 (cointegration), models (9.1) to (9.3) and (10.1) to (10.3), with several parameter values. For comparison, the numerical values assigned for β_1, β_2, β_3 and of δ, are the same for all tests. From Table 3 to Table 14, we summarize the power of these test, under different parameter values of β_1, β_2, β_3 and of δ, in equations (9.1) to (9.3) and the nonlinear graph are in the following Figure 3,

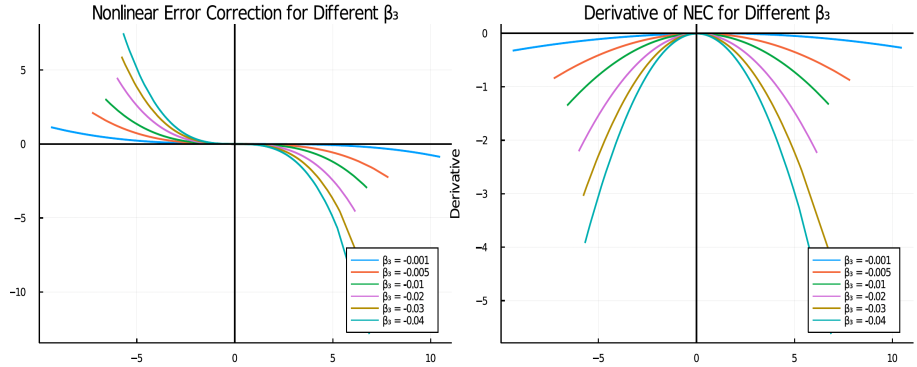

Figure 3. NEC-Cubic Polynomials and their first derivatives for the different parameter values used in the Monte Carlo analysis of the power of NEC-test and NEG-test

From the analysis of the power of the EC-test and the NEC-test, included from Table 3 to Table 8, we take the following conclusions: i) When de DGP is a linear EC model (9.1), the highest power is obtained with the t-ratio (t^*_3), from equation (9.2). The order of the power of the test statistics is the following, $t^*_3 > t^*_1 > t_1 > t_3$. ii) When the DGP is NEC(3) model (9.2), again t^*_3 has the maximum power but t_3 has high power as well. iii) When the DGP is NEC(1,2,3) model (9.3), t^*_3 has the maximum power, t_3 has high power as well but t_1 has low power. In summary, the simplest and more powerful error correction test, is t^*_3 from NEC(3) model (9.2), independent on whether the DGP is linear a linear EC or is a nonlinear EC, NEC. In summary, t^*_3, t-ratio from the pure cubic model NEC(3), is the most powerful error correction test, no matter whether the model is linear or nonlinear.

From the analysis of the power of the EG-test and the NEG-test, included from Table 9 to Table 14, we obtain the following conclusions: i) When the DGP is the linear EC model (10.1), the most powerful test is the t-ratio (t_1) from the NEC model (10.3). The order of the power in these cases is $t_1 > t^*_1 > t^*_3 > t_3$. ii) When the DGP is NEC(3) model (9.2), the highest power is from t^*_3 in model (10.2), followed by t_3 in model (10.3). iii) When the DGP is NEC(1,2,3) model (10.3), the three test statistic has good power but the order is $t_1 > t^*_3 > t_3 > t^*_1$. In summary, when testing for cointegration using the residual estimated in the first-step of Engle and Granger (1987) approach, we recommend to start with the t-ratio of the pure NEC(3), t^*_3 , and if you do not reject the H of no cointegration, use the t-ratio, t_1, obtained by estimating the full cubic polynomial, NEC(1,2,3).

For the analysis of the power of the NEC-test and the NEG-test we have the results are included from Table 3 to Table 8.

Table 3. Power of t_1

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$	$\beta_1=-0.1$	$\beta_1=-0.2$
(1)	150	0.0	0.050	0.046	0.142	0.402	0.841
	150	0.5	0.050	0.050	0.108	0.270	0.667
	150	1.0	0.050	0.049	0.092	0.236	0.595
(1)	500	0.0	0.049	0.046	0.702	0.979	1.000
	500	0.5	0.049	0.042	0.499	0.902	0.998
	500	1.0	0.050	0.042	0.422	0.842	0.993
(2)	150	0.0	0.092	0.096	0.194	0.430	0.842
	150	0.5	0.084	0.082	0.121	0.288	0.669
	150	1.0	0.059	0.060	0.111	0.248	0.586
(2)	500	0.0	0.248	0.264	0.676	0.970	1.000
	500	0.5	0.114	0.122	0.476	0.871	0.998
	500	1.0	0.098	0.100	0.410	0.818	0.994

DGP (1): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$
DGP (2): equations (6.1)-(6.3) with $\beta_2, \beta_3 \neq 0$

Table 4. Power of t_3

DGP	T	δ	$\beta_3=-0.001$	$\beta_3=-0.003$	$\beta_3=-0.05$
(1)	150	0.0	0.084	0.232	0.938
	150	0.5	0.070	0.138	0.866
	150	1.0	0.060	0.110	0.758
(1)	500	0.0	0.356	0.796	0.896
	500	0.5	0.254	0.546	0.997
	500	1.0	0.208	0.430	0.998
(2)	150	0.0	0.080	0.232	0.944
	150	0.5	0.069	0.138	0.867
	150	1.0	0.056	0.110	0.756
(2)	500	0.0	0.306	0.796	0.908
	500	0.5	0.244	0.545	0.996
	500	1.0	0.199	0.430	0.998

DGP (1): equations (6.1)-(6.3) with $\beta_1=\beta_2=0$
DGP (2): equations (6.1)-(6.3) with $\beta_1, \beta_2 \neq 0$

Table 5. Power of t_3

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$
(3)	150	0.0	0.052	0.048	0.030
	150	0.5	0.050	0.048	0.041
	150	1.0	0.048	0.048	0.036
(3)	500	0.0	0.052	0.050	0.027
	500	0.5	0.054	0.050	0.032
	500	1.0	0.054	0.051	0.026

DGP(3): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$

Table 6. Power of t_1^*

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$	$\beta_1=-0.1$	$\beta_1=-0.2$
(1)	150	0.0	0.047	0.052	0.294	0.832	1.000
	150	0.5	0.048	0.050	0.198	0.612	0.992
	150	1.0	0.047	0.053	0.174	0.507	0.977
(1)	500	0.0	0.052	0.058	0.996	1.000	1.000
	500	0.5	0.052	0.059	0.964	1.000	1.000
	500	1.0	0.058	0.060	0.916	1.000	1.000
(2)	150	0.0	0.200	0.210	0.610	0.934	1.000
	150	0.5	0.129	0.132	0.324	0.726	0.997
	150	1.0	0.114	0.114	0.249	0.593	0.985
(2)	500	0.0	0.901	0.932	1.000	1.000	1.000
	500	0.5	0.734	0.774	1.000	1.000	1.000
	500	1.0	0.492	0.526	0.998	1.000	1.000

DGP (1): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$
DGP (2): equations (6.1)-(6.3) with $\beta_2, \beta_3 \neq 0$

Table 7. Power of t_3^*

DGP	T	δ	$\beta_3=-0.001$	$\beta_3=-0.003$	$\beta_3=-0.05$	$\beta_3=-0.1$	$\beta_3=-0.2$
(1)	150	0.0	0.284	0.720	1.000	1.000	1.000
	150	0.5	0.150	0.388	1.000	1.000	1.000
	150	1.0	0.131	0.256	1.000	1.000	1.000
(1)	500	0.0	0.977	1.000	1.000	1.000	1.000
	500	0.5	0.857	0.996	1.000	1.000	1.000
	500	1.0	0.712	0.990	1.000	1.000	1.000
(2)	150	0.0	0.242	0.720	1.000	1.000	1.000
	150	0.5	0.142	0.386	1.000	1.000	1.000
	150	1.0	0.128	0.256	1.000	1.000	1.000
(2)	500	0.0	0.900	1.000	1.000	1.000	1.000
	500	0.5	0.813	0.996	1.000	1.000	1.000
	500	1.0	0.672	0.990	1.000	1.000	1.000

DGP (1): equations (6.1)-(6.3) with $\beta_1=\beta_2=0$
DGP (2): equations (6.1)-(6.3) with $\beta_1, \beta_2 \neq 0$

Table 8. Power of t_3^*

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$
(3)	150	0.0	0.084	0.232	0.938
	150	0.5	0.070	0.138	0.866
	150	1.0	0.060	0.110	0.758
(3)	500	0.0	0.356	0.796	0.896
	500	0.5	0.254	0.546	0.997
	500	1.0	0.208	0.430	0.998

DGP (3): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$

For the analysis of the power of the NEC-test and the NEG-test we have the results included from Table 9 to Table 14.

Table 9. Power of t_1

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$	$\beta_1=-0.1$	$\beta_1=-0.2$
	150	0.0	0.521	0.531	0.728	0.931	0.988
(1)	150	0.5	0.524	0.538	0.806	0.968	0.994
	150	1.0	0.526	0.541	0.838	0.972	0.998
	500	0.0	0.591	0.584	0.999	1.000	1.000
(1)	500	0.5	0.594	0.609	1.000	1.000	1.000
	500	1.0	0.599	0.632	1.000	1.000	1.000
	150	0.0	0.687	0.697	0.893	0.962	0.992
(2)	150	0.5	0.741	0.746	0.921	0.979	0.996
	150	1.0	0.749	0.756	0.929	0.986	1.000
	500	0.0	0.936	0.954	1.000	1.000	1.000
(2)	500	0.5	0.994	0.996	1.000	1.000	1.000
	500	1.0	0.996	0.998	1.000	1.000	1.000

DGP (1): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$
DGP (2): equations (6.1)-(6.3) with $\beta_2,\beta_3\neq 0$

Table 10. Power of t_3

DGP	T	δ	$\beta_3=-0.001$	$\beta_3=-0.003$	$\beta_3=-0.05$
	150	0.0	0.066	0.126	0.862
(1)	150	0.5	0.063	0.098	0.778
	150	1.0	0.054	0.086	0.723
	500	0.0	0.198	0.505	0.895
(1)	500	0.5	0.175	0.413	0.996
	500	1.0	0.165	0.373	0.998
	150	0.0	0.064	0.126	0.868
(2)	150	0.5	0.064	0.098	0.778
	150	1.0	0.055	0.086	0.722
	500	0.0	0.168	0.504	0.906
(2)	500	0.5	0.168	0.414	0.996
	500	1.0	0.158	0.373	0.998

DGP (1): equations (6.1)-(6.3) with $\beta_1=\beta_2=0$
DGP (2): equations (6.1)-(6.3) with $\beta_1,\beta_2\neq 0$

Table 11. Power of t_3

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$
	150	0.0	0.048	0.046	0.044
(3)	150	0.5	0.050	0.046	0.039
	150	1.0	0.048	0.049	0.034
	500	0.0	0.054	0.048	0.029
(3)	500	0.5	0.048	0.044	0.028
	500	1.0	0.052	0.045	0.019

DGP(3): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$

Table 12. Power of t_1^*

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$	$\beta_1=-0.1$	$\beta_1=-0.2$
	150	0.0	0.052	0.050	0.102	0.263	0.730
(1)	150	0.5	0.052	0.050	0.138	0.360	0.844
	150	1.0	0.051	0.055	0.153	0.404	0.866
	500	0.0	0.045	0.048	0.668	0.985	1.0
(1)	500	0.5	0.043	0.048	0.762	0.996	1.0
	500	1.0	0.049	0.048	0.820	0.996	1.0
	150	0.0	0.082	0.082	0.156	0.363	0.760
(2)	150	0.5	0.104	0.104	0.189	0.438	0.866
	150	1.0	0.100	0.102	0.198	0.472	0.880
	500	0.0	0.428	0.466	0.944	0.997	1.0
(2)	500	0.5	0.374	0.404	0.964	0.998	1.0
	500	1.0	0.350	0.378	0.966	0.999	1.0

DGP (1): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$
DGP (2): equations (6.1)-(6.3) with $\beta_2,\beta_3\neq 0$

Table 13. Power of t_3^*

DGP	T	δ	$\beta_3=-0.001$	$\beta_3=-0.003$	$\beta_3=-0.05$
	150	0.0	0.083	0.220	0.929
(1)	150	0.5	0.081	0.174	0.968
	150	1.0	0.093	0.174	0.962
	500	0.0	0.634	0.956	0.884
(1)	500	0.5	0.586	0.944	0.996
	500	1.0	0.549	0.938	1.000
	150	0.0	0.064	0.126	0.868
(2)	150	0.5	0.064	0.098	0.778
	150	1.0	0.055	0.086	0.722
	500	0.0	0.168	0.504	0.906
(2)	500	0.5	0.168	0.414	0.996
	500	1.0	0.158	0.373	0.998

DGP (1): equations (6.1)-(6.3) with $\beta_1=\beta_2=0$
DGP (2): equations (6.1)-(6.3) with $\beta_1,\beta_2\neq 0$

Table 14. Power of t_3^*

DGP	T	δ	$\beta_1=-0.001$	$\beta_1=-0.003$	$\beta_1=-0.05$
	150	0.0	0.054	0.059	0.086
(3)	150	0.5	0.051	0.056	0.090
	150	1.0	0.052	0.052	0.115
	500	0.0	0.049	0.053	0.410
(3)	500	0.5	0.048	0.050	0.490
	500	1.0	0.051	0.054	0.515

DGP (3): equations (6.1)-(6.3) with $\beta_2=\beta_3=0$

5. NONLINEAR MODELING OF MONEY DEMAND

5.1. Nonlinear Long-Run Money Demand

Based on the QTM, it is interesting is to address the question of which is the economic variable that moves together with velocity in the long-run (cointegration), so that after controlling for this variable, the monetary aggregate grow at the same rate as nominal output. The purpose of the subsection is to re-evaluate empirically this nonlinear cointegrated money demand equation with the new available information until year 2023.

The cointegration relationship between the inverse velocity of money $(m - p - y)_t$ and the opportunity cost measure RNA_t is remarkable, as illustrated in Figure 1.C. This is supported with the cointegration tests done in the NEC representation estimated later. This translates that any permanent increase in the opportunity cost measure has a permanent increase in velocity and therefore a decrease in real money balances. Further, any deviation of the two series from their long-run relationship is transitory and constrained to disappear in the long run. To formally examine cointegration, we apply several cointegration tests: conventional tests (Engle-Granger and Johansen cointegration test), and linear and nonlinear cointegration tests derived previously from Granger's Representation Theorem for NEC models to capture potential nonlinearities that standard methods may ignore.

Before proceeding with the subsequent analysis of the inverse velocity of money and short-term interest rates, we verified the existence of a single cointegrating vector between the long-run equilibrium determinants of the QTM, $X_t = \{m_t, p_t, y_t, rs_t\}$. This is supported at 5% level of significance by the Johansen Unrestricted Cointegration Rank Test (Trace) statistic of a VECM(2) with a long-term constant yielded 72.31 for the "none" hypothesis and 33.39 for "at most one" cointegrating relationship. The impulse response analysis of this system is evaluated in the policy implications (Section 6).

Moreover, results support the unit income and price elasticity in the long run (β_y= 1.00 and β_p= 1.00). Notice that this condition rather than restrictive, it reduces the standard error of the regression and is aligned theoretical models regarding the liquidity preference function to be equal to unity in the long term when market frictions are no longer binding. Under these conditions, the difference between broad real money and nominal output is driven by the short-run interest rate (RNA_t). Therefore, in the long-run velocity of circulation of money (v_t) moves together with short-term interest rates.

The results of the cointegration tests are presented in Table 15. Overall, cointegration is strongly supported between the inverse velocity of money $(m - p - y)_t$ and the opportunity cost measures (RS_t and RNA_t) in both levels and log specification. Notably, the Johansen test consistently identifies a single cointegration vector, confirming the suitability of a single-equation empirical modeling approach for money demand.

While the standard Engle-Granger test detects cointegration at the 10% significance level only for the *log-level* specification with RS_t, Johansen cointegration test identifies a single cointegration vector across all cases at the same significance level. Furthermore, the previously derived parametric cointegration tests reinforce that that $(m - p - y)_t$ is *nonlinearly cointegrated* with the opportunity cost measure (RNA_t, RS_t) in a *Log-Level* form. As outlined earlier, when applying linear and nonlinear EG-tests and the NEG-test using residuals from the first step of the Engle-Granger (1987) approach, we recommend starting with the t-ratio of the NEG (t_3^*), following this order $t^*_3 > t_1 > t_3 > t^*_1$.Thus, if the null hypothesis of no cointegration is not rejected, the next step should involve the t-ratio NEG (t_1), obtained from the full cubic polynomial NEC(1,2,3) model and so on. Following this procedure, we reject satisfactorily the null hypothesis of no cointegration at 10% level for all cases, except for nominal interest rates in logs (rs_t).

Table 15. Cointegration Tests (1877 – 2023)

	Linear Cointegration Tests			*Nonlinear Cointegration Tests*		
Variables	***EG***	***EG*** (t_1^*)	***Johansen***	***NEG*** (t_1)	***NEG*** (t_3)	***NEG*** (t_3^*)
$\{(m-p-y)_t\,;RS_t\}$	-3.12*	-2.53	17.01** (1)	0.07	-2.47*	-3.03
$\{(m-p-y)_t\,;rs_t\}$	-2.64	-2.91	16.60** (1)	-0.72	-1.61	-2.55
$\{(m-p-y)_t\,;RNA_t\}$	-2.23	-2.46	13.92* (1)	-0.88	-2.20*	-2.37
$\{(m-p-y)_t\,;rna_t\}$	-3.03	-3.19*	16.11** (1)	-3.91*	1.88	-0.02

Notes: Each cell presents the corresponding test-statistic values of the null hypothesis of no-cointegration (unit root) with and the level of rejection in asterisks (** if significant at 5%, * if significant at 10%): (i) Engle-Granger (EG) tau-statistic from *Eviews 13*; (ii) Linear EG test *EG*(t_1^*) (iii) Johansen´s cointegration trace-statistic of a VECM(2), with dual constant term between $(m-p-y)_t$ and short-term interest rate measure, suggests (in parenthesis) the number of cointegration relationships at 10% level; (iv) Nonlinear EG test *NEG*(t_1), (v) Nonlinear EG test *NEG*(t_3) and (vi) Nonlinear EG test *NEG*(t^*_3). The 5% and 10% critical values (c.v.) for rejecting the null hypothesis of no cointegration are provided in Table 2, based on c.v. specifically generated for a sample size of T = 150. Source: Authors´ calculations.

It is important to note that while these tests implicitly impose COMFAC restrictions, more general and robust analog tests, such as the EC-test and NEC-test can be conducted within an error-correction framework. In this context, these tests are applied to the *cubic-polynomial equilibrium correction* model, Model A (see Table 19), for the demand of real money balances. The NEC (t_3^*) and NEC (t_3) tests confirm the presence of a strong nonlinear equilibrium-correction relationship at the 5% significance level. Three nonparametric cointegration tests were also employed to address potential nonlinearities: the Record counting cointegration (RCC) test (Escribano, Sipols, and Aparicio 2006b), Induced-order cointegration based on Kolmogorov-Smirnov (KS) test (Escribano, Santos, and Sipols 2008) and Cramer-Von Misses (CVM) cointegration test (Escribano, Santos, and Sipols 2018). In all the cases, we reject the null hypothesis of no cointegration, see Escribano and Rodríguez (2023).

Alternative cointegration relationships for UK money demand could be estimated using various functional forms (see Escribano and Rodriguez, 2023). Conventionally this relationship is specified as *log-level* functional form due to its theoretical and empirical implications. This specification relates to the theoretical framework described by Benati et al. (2021), where the equation is derived from a representative agent economy with labor as the only input, costly transactions (as in the Baumol-Tobin model), and exogenous stochastic productivity. Moreover, the *log-level* specification offers a key advantage: it allows for negative or near-zero interest rates (liquidity-trap scenarios), making it suitable for both empirical estimation and policy analysis.

The inverse velocity (long-run money demand) and the opportunity cost measure (i.e. interest rates) are *nonlinearly cointegrated* in Equation (11.1) and (11.2) when u_{Rt} is I(0). Using traditional short-term interest rates (RS_t), the Fully-Modified OLS (FM-OLS, See Phillips and Hansen (1990)) estimates are as follows:

$$(m-p-y)_t = \underset{(0.043)}{-0.29} - \underset{(0.79)}{4.92}\, RS_t + u_{Rt} \qquad (11.1)$$

$T = 147$ (1877 - 2023) $\quad R^2 = 0.46 \quad 100 * \hat{\sigma} = 16.92\%$

As argued by Ericsson, Hendry, and Prestwich (1998), the empirical performance of this cointegration relationship can be further improved by incorporating an adjusted opportunity cost measure, RNA_t, as in Equation (11.2):

$$(m-p-y)_t = \underset{(0.032)}{-0.26} - \underset{(0.82)}{8.11}\, RNA_t + u_{Rt} \qquad (11.2)$$

$T = 147$ (1877 - 2023) $\qquad R^2 = 0.63 \qquad 100 * \hat{\sigma} = 13.91\%$

The negative and significant semi-elasticity with respect to RNA_t is aligned with the general predictions of the literature and plausible from the theoretical viewpoint. Equation (11.2) provides a more robust fit for the observed data and will serve as the foundation for the subsequent sections of this analysis.

5.2. Nonlinear Equilibrium Correction Specifications

We know by Granger's representation theorem that if the variables are cointegrated there is an error-correction representation, in at least one of the equations of the system. Under linear cointegration the error-correction adjustment can be linear (Engle and Granger 1987; Johansen 1992) or nonlinear (Escribano 1985; 1986; 1987; 2004; Escribano and Mira 2002, Saikkonen 2005; Escanciano and Escribano 2009; Kapetanios, Shin, and Snell 2006; Teräsvirta, Tjostheim, and Granger 2010; Kilic 2010, Hwan Seo 2011). Similarly, under nonlinear cointegration the error-correction model could be linear or nonlinear (Escribano 1986; 1985; 2004; Escribano and Pfann 1998; Escribano and Granger 1998; Chang, Park and Phillips 2001; Saikkonen 2005; Saikkonen and Choi 2004; Tjostheim 2020).

Focusing on the exponential cointegration relationship expressed in *log-level* form, Equation (11.1) and (11.2), we consider the error term u_{rt}, stationary or I(0). The underlying concept is as follows, let $u_{rt} = \rho\, u_{rt-1} + \omega_t$ with $\rho < 1$ and $\omega_t \sim I(0)$ but not necessarily white noise,

$$-v_t = \beta_{R0} + \beta_{R1} R_t + u_{Rt} \qquad (12a)$$

$$\Delta u_{Rt} = (\rho - 1)\, u_{Rt-1} + \omega_t \quad with\ (\rho - 1) < 0. \qquad (12b)$$

A *cubic polynomial error-correction adjustment*, with the COMFAC restriction imposed, can be estimated in two stages. First, we estimate equation (13) and use those residuals to estimate a nonlinear version of Engle and Granger (1987) *test for non-cointegration* in the second equation (14) testing the null hypothesis of no-cointegration; $H_0: \rho_1 = \rho_2 = \rho_3 = 0$,

$$-v_t = \hat{\beta}_{R0} + \hat{\beta}_{R1} R_t + \hat{u}_{Rt} \qquad (13)$$

$$\Delta\hat{u}_{Rt} = \rho_1 \hat{u}_{Rt-1} + \rho_2 \hat{u}^2_{Rt-1} + \rho_3 \hat{u}^3_{Rt-1} + \omega_t \tag{14}$$

A generalization of (14) with a general non*linear equilibrium-correction model with exponential cointegration* with dynamics for the rate of growth of the real balances, rate of growth of real income, inflation and changes in interest rates, without imposing the COMFAC restriction, and including a vector ΔX_t of additional relevant control I(0) variables, like the long-term interest rates in rates, and the inflation rate , etc., are given in Equations (15a) and (15b),

$$-v_t = \beta_{R0} + \beta_{R1} R_t + u_{Rt} \tag{15a}$$

$$\Delta(m-p)_t = \gamma_0 + \phi_m(L)\Delta(m-p)_{t-1} + \phi_y(L)\Delta y_t + \phi_p(L)p_t + \\ + \gamma_{r1}(L)\Delta R_t + \alpha'(L)\Delta X_t + \rho_{R1} u_{t-1} + \rho_{R2} u^2_{t-1} + \rho_{R3} u^3_{t-1} + \varepsilon_{vt}. \tag{15b}$$

where the errors terms (ε_{vt}) of Equation (15) are white noise, $\phi_m(L), \phi_y(L), \phi_p(L), \gamma_{r1}(L), \gamma_{r2}(L), \gamma_{r3}(L)$ and $\alpha'(L)$ are all finite order polynomials in the lag operator L of maybe different orders, with all the roots outside the unit circle, and where, for large values of $|u_{rt-1}|$, $\rho_1\, u_{rt-1} + \rho_2\, u^2_{rt-1} + \rho_3\, u^3_{rt-1}$ is dominated by a linear function of u_{rt-1} with negative slope (error-correcting), see Escribano (2004), Escribano and Mira (2002), and Saikkonen (2005). Equation (15b) could have errors terms (ε_{vt}) having ARCH type heteroskedasticity as in Saikkonen (2005).

In the literature on empirical applications of nonlinear error-correction models, this nonlinear functions have been estimated using different procedures; parametrically using *cubic-polynomial* equilibrium-correction specifications as in Escribano (1985, 1986, 1987, 2004) and in Hendry and Ericsson (1991), or Logistic Smooth Transition (STR) equilibrium-correction specifications (Teräsvirta and Eliasson 2001; Kapetanios, Shin, and Snell 2006; Kilic 2010 Teräsvirta, Tjostheim, and Granger 2010; Escribano and Torrado 2018), or by machine learning methods, like a Random Forest nonlinear equilibrium-correction specification, as was done in Escribano and Wang (2021).

To make valid inferences when estimating NEC models, by OLS or by NLS, in a two-steps, it is important to validate De Jong (2001)`s orthogonality conditions. De Jong (2001) proved that while the super-consistency of the OLS and/or NLS estimators in the first step is beneficial, it does not guarantee the invariance of parameter estimates in the second step, unless certain orthogonality conditions are satisfied[6]. Escribano and Rodríguez (2023), verified that these orthogonality conditions are satisfied for the UK and the US money demands. When they are not satisfied, we may need to consider joint estimation techniques of the long-run and short-run parameters by say NLS, which in the case of a cubic polynomial or STR models might not that simple.

5.3. Nonlinear Money Demand Equations (1878 – 2023)

The goal of this section is to, review and extend the estimation of competing UK real money demand models until 2023. We will evaluate their forecasting performance and their parameter stability, which are crucial issues for the use of money as an instrument in monetary policy.

5.4. Previous Estimates and their Mechanistic Extension

Money demand relationships have been a critical and persistent point for economists for over half a century, as highlighted in Table B.1. The development of nonlinear (cubic-polynomial) error-correction models by Escribano (1985, 1986, 1987), together with a few suggestions from Longbottom and Holly (1985), allowed Hendry and Ericsson (1991) to produce a better equilibrium-correction specification of the UK money demand, based on cubic polynomials. Afterwards, Ericsson, Hendry, and Prestwich (1998) extended the analysis until 1993, incorporating an interesting measure of opportunity cost (RNA_t). Subsequently, Teräsvirta and Eliasson (2001) refined these previous models of the demand for broad money in the UK by means of smooth transition regression (STR). Similarly, Escribano (2004) encompassed the analysis to new parametric adjustments (cubic polynomials and rational polynomials), and semiparametric (smoothing splines) NEC models with stable results for the extended sample period considered, from 1878 to 2000.

In particular, the identification of the oportunity cost measures such as $RN_{t,}$, introduced by Friedman and its refined version, RNA_t, proposed later by Ericsson, Hendry and Prestwitch (1998) is crucial to address the mispredictions inherent in historical spliced monetary aggregates, as discussed in Escribano and Rodriguez (2023). In this context, we focus on the most recent empirical models in the literature that

[6] In general, those conditions are satisfied if we include a constant term when we estimate the cointegrating equation in the first step.

incorporate this contribution—specifically, the models of Ericsson, Hendry, and Prestwich (1998), Teräsvirta and Eliasson (2001) and Escribano (2004).

The replication and mechanic extension of these empirical competing models to 2023, as presented in Table 16, reveal noteworthy findings. The estimated equations appear to be congruent and well specified within the sample (see Figure 4), given the diagnostic statistics, even after extending mechanically the sample period by more than 20 years. This highlights the robustness and significance of their original specification.

For the mechanical extension of the model up to 2023, two periods of excess money demand were identified. The first period corresponds to the financial crisis ($DCRISIS_t$). This dummy takes a value of 1 from 2000 to 2007, -1 from 2008 to 2011, and 0 otherwise. It reflects a period of excess money demand before the subprime mortgage crisis, which was subsequently corrected by an equivalent excess reduction in the years following the crisis. The second period of excess demand is associated with the COVID-19 pandemic in 2020, represented by the dummy variable $DCOVID_t$, which takes a value of 1 in 2020 and 0 otherwise. This excess demand impacted the economy with a two-year lag, manifesting in 2022. During this period, the relaxation of economic restrictions imposed to control the pandemic was not accompanied by a proportional increase in money demand or any other variables within the model. Adapting the model to new economic environments is essential. Therefore, the inclusion of these exogenous dummy variables is necessary to ensure the robustness and coherence of the models when extending them over time.

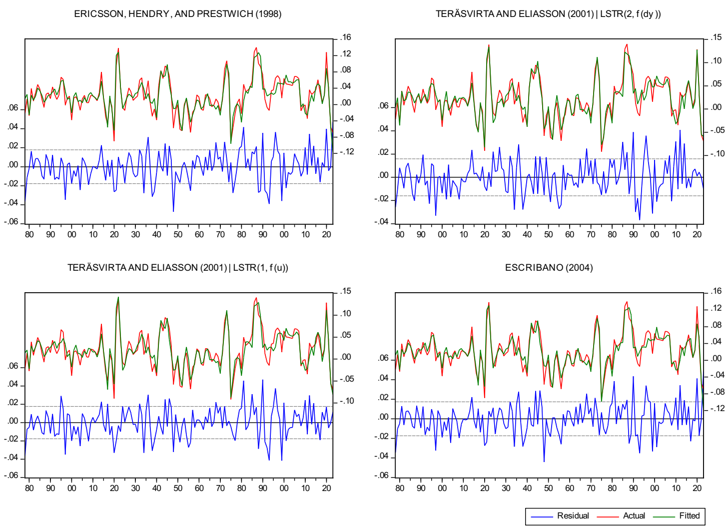

Figure 4. UK Estimated Extended Nonlinear Error-Correction Models (1878 – 2023)

Notes: Figure plots the actual values $\Delta(m-p)_t$, fitted values $\Delta(\widehat{m-p})_t$ and residuals $\hat{\varepsilon}_{vt}$ of extended error-correction models presented in Table 16. Source: Authors' calculations.

Table 16. UK Competing Mechanical Extended Money Demand Models (1878 – 2023)

	Dependent variable: $\Delta(m-p)_t$							
Regressors	Ericsson, Hendry, and Prestwich (1998)		Teräsvirta and Eliasson (2001) *LSTR(1,* $f(\hat{u}_{Rt-1})$*)*		Teräsvirta and Eliasson (2001) *LSTR(2,* $f(\Delta y_t)$*)*		Escribano (2004)	
	Original	*Extended*	*Original*	*Extended*	*Original*	*Extended*	*Original*	*Extended*
Linear components								
$\Delta(m-p)_{t-1}$	0.48 (0.07)	0.47 (0.051)	0.52 (0.051)	0.52 (0.057)	0.83 (0.10)	0.81 (0.11)	0.67 (0.078)	0.52 (0.070)
$\Delta(m-p)_{t-2}$	-	-	-0.15 (0.040)	-0.12 (0.049)	-0.16 (0.039)	-0.15 (0.040)	-0.24 (0.063)	-0.11 (0.060)
$\Delta^2(m-p)_{t-2}$	-0.10 (0.04)	-0.09 (0.041)	-	-	-	-	-0.09 (0.042)	-0.059 (0.043)
Δp_t	-0.62 (0.07)	-0.62 (0.044)	-0.66 (0.044)	-1.05 (0.21)	-0.62 (0.040)	-0.63 (0.044)	-0.64 (0.042)	-0.62 (0.045)
Δp_{t-1}	0.40 (0.07)	0.43 (0.049)	0.46 (0.042)	0.45 (0.049)	0.64 (0.069)	0.66 (0.099)	0.58 (0.077)	0.47 (0.075)
Δp_{t-2}	-	-	-	-	-	-	-0.15 (0.059)	-0.044 (0.060)
Δrna_t	-0.020 (0.006)	-0.014 (0.0052)	-0.022 (0.005)	-0.012 (0.005)	-0.015 (0.005)	-0.013 (0.005)	-0.02 (0.005)	-0.015 (0.005)
$\Delta_2 rl_t$	-0.041 (0.019)	-0.038 (0.0076)	-	-	-	-	-0.040 (0.013)	-0.036 (0.008)
$\Delta_2 rl_{t-2}$	-	-	-	-	-	-	-0.030 (0.015)	-0.015 (0.012)
$(D1+D3)_t$	0.039 (0.006)	0.033 (0.0057)	0.037 (0.005)	0.030 (0.006)	0.035 (0.005)	0.034 (0.006)	0.030 (0.005)	0.033 (0.0058)
$D4_t * \Delta rs_t$	0.10 (0.042)	0.083 (0.029)	0.07 (0.024)	0.086 (0.031)	0.12 (0.022)	0.10 (0.027)	0.080 (0.027)	0.071 (0.028)
DC_t	0.052 (0.010)	0.059 (0.0074)	0.061 (0.007)	0.062 (0.008)	0.071 (0.007)	0.068 (0.007)	0.050 (0.008)	0.061 (0.008)
$DCRISIS_t$	-	0.028 (0.005)	-	0.025 (0.006)	-	0.029 (0.005)	-	0.028 (0.005)
$DCOVID_{t-2}$	-	0.09 (0.024)	-	0.074 (0.010)	-	0.15 (0.026)	-	0.085 (0.027)
$\hat{u}_{Rt-1}$	-	-	-0.17 (0.022)	-0.017 (0.057)	0.092 (0.046)	0.11 (0.07)	-	-
Constant	0.004 (0.002)	0.007 (0.002)	-	-	-	-	0.008 (0.002)	0.009 (0.002)
Nonlinear components								
$(\hat{u}_{Rt-1}-0.2)\hat{u}_{Rt-1}{}^2$	-2.26 (0.46)	-1.42 (0.21)	-	-	-	-	-2.02 (0.38)	-1.34 (0.21)
$f(*) * \Delta(m-p)_{t-1}$	-	-	-	-	-0.49 (0.11)	-0.46 (0.13)	-	-
$f(*) * \hat{u}_{Rt-1}$	-	-	-0.62 (0.23)	-0.019 (0.059)	-0.19 (0.048)	-0.21 (0.071)	-	-
$f(*) * \Delta p_t$	-	-	0.38 (0.18)	0.43 (0.21)	-	-	-	-
$f(*) * \Delta p_{t-1}$	-	-	-	-	-0.34 (0.065)	-0.35 (0.10)		
$f(*) * \Delta_2 rl_t$	-	-	-0.24 (0.081)	-0.059 (0.009)	-0.072 (0.018)	-0.078 (0.009)	-	-
$f(*) * Constant$	-	-	0.21 (0.007)	0.011 (0.002)	0.016 (0.002)	0.016 (0.003)	-	-
Sample	1878-1993	1878–2023	1878-1993	1878–2023	1878-1993	1878–2023	1878-2000	1878–2023
R^2	0.87	0.84	0.90	0.85	0.90	0.88	0.87	0.85
$100 * \hat{\sigma}$	1.62%	1.77%	1.48%	1.77%	1.43%	1.59%	1.61%	1.74%
Misspecification test	***P-value***		***P-value***		***P-value***		***P-value***	
Autocorrelation (2)	0.17	0.73	0.43	0.61	0.94	0.95	0.15	0.51
RESET (1)	0.31	0.73	-	-	-	-	0.43	0.17
ARCH (1)	0.36	0.30	0.44	0.63	0.96	0.58	0.62	0.19
Huber-White	0.57	0.09	0.56	0.80	0.68	0.63	0.21	0.11
Normality	0.83	0.79	0.19	0.89	0.94	0.70	0.91	0.46

Notes: Each column of the short-run equations presents coefficients obtained from separate regressions, with standard errors provided in parentheses. The original models' results are retrieved from papers cited above. All reported statistical tests are expressed as p-values. Below are the details of the models:

(i) All extended models used two-step cointegrating residuals by FM-OLS: $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.26 + 8.12\ RNA_{t-1}$. Original models employed following residuals: $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.34 + 6.30\ RNA_{t-1}$ for Ericsson, Hendry, and Prestwich (1998), $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.32 + 6.67\ RNA_{t-1}$ for Teräsvirta and Eliasson (2001) and $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.35 + 6.16\ RNA_{t-1}$ for Escribano (2004).

(ii) Teräsvirta and Eliasson (2001)`s LSTR(1) with transition function $f(\hat{u}_{Rt-1}) = \{1+\exp[\gamma\ (\hat{u}_{Rt-1} - c)/\hat{\sigma}_{\hat{u}_{Rt-1}}]\}^{-1}$, estimated original model parameters by NLS are $\gamma = -2.54$ and $c = 0.19$ and extended parameters are $\gamma = -9.26$ and $c = 0.27$.

(iii) Teräsvirta and Eliasson (2001)`s LSTR(2) with transition function $f(\Delta y_t) = \{1-\exp[\gamma\ (\Delta y_t - c_1)((\Delta y_t - c_2)/\hat{\sigma}^2_{\Delta y_t}]\}^{-1}$, estimated original model parameters by NLS are $\gamma = -2.68$, $c_1 = 0.011$ and $c_2 = 0.052$ and extended parameters are $\gamma = -1.67$, $c_1 = 0.010$ and $c_2 = -0.046$.

5.5. *Selected Nonlinear Equilibrium-Correction (NEC) Models for the Real Money Demand of UK*

While it is true that extended model's performance is fairly suitable, purely mechanical extensions can result in incoherent economic representations and lead to predictive failures (Ericsson, Hendry, and Prestwich 1998). In this context, we empirically re-evaluate these extended nonlinear cointegrated money demand equations and propose selected versions that incorporate newly available information up to 2023.

The selected model versions are summarized in Table 17. Model A is a revaluation of the NEC models proposed by Ericsson, Hendry, and Prestwich (1998) and Escribano (2004), incorporating a nonlinear adjustment toward equilibrium through a flexible polynomial specification (*NEC – Cubic polynomial*). Models B and C correspond to the smooth transition regressions (STR) of Teräsvirta and Eliasson (2001). Model B utilizes a *first-order logistic smooth transition* specification, with the nonlinear equilibrium-correction term as the transition variable (*NEC - LSTR(1,* $f(\hat{u}_{Rt-1})$)). In contrast, Model C captures the nonlinear dynamics through a *second-order logistic smooth transition* regression, using the growth rate of real income as the transition variable (*EC - LSTR(2,* $f(\Delta y_t)$). Finally, one might hypothesize the existence of a money demand equation that integrates the nonlinear characteristics of Model A and C into a single framework, that is Model D: a *second-order logistic smooth transition regression* model, using the growth rate of real income as the transition variable (*LSTR(2,* $f(\Delta y_t)$*)*, combined with nonlinear equilibrium specfied as a *cubic polynomial* error-correction term.

STR models are estimated using nonlinear least squares (NLS), a method sensitive to initial conditions due to its nonlinear optimization nature. To ensure convergence, initial parameters are obtained from the original models of Teräsvirta and Eliasson (2001). In contrast, the cubic-polynomial specification in Model A, estimated via ordinary least squares (OLS), avoids this estimation concern, requires fewer parameters, and allows for direct interpretation of coefficients.

All selected equations exhibit consistency with plausible coefficient signs and magnitudes and are well-specified within the sample (see Figure 5), as supported by diagnostic statistics and subsequent parameter constancy tests. Furthermore, the three selected empirical models exhibit a high goodness-of-fit to the historical demand for real money balances and explain over 85% of the variability in the data, surpassing the in-sample performance of their extended model counterparts.

Table 17. Selected Nonlinear Money Demand Models of the UK (1878 – 2023)

	Dependent variable: $\Delta(m-p)_t$			
Regressors	*(a) NEC - Cubic pol.*	*(b) NEC - LSTR(1)*	*(c) EC - LSTR(2)*	*(d) NEC - LSTR(2)*
Linear components				
$\Delta(m-p)_{t-1}$	0.49 (0.049)	0.52 (0.051)	0.75 (0.089)	0.75 (0.095)
$\Delta(m-p)_{t-2}$	-0.17 (0.041)	-0.21 (0.042)	-0.17 (0.040)	-0.15 (0.038)
$\Delta^2 p_t$	-0.58 (0.042)	-0.67 (0.065)	-0.62 (0.046)	-0.61 (0.043)
Δp_{t-1}	-0.16 (0.034)	-0.16 (0.036)	- -	- -
Δrna_t	-0.016 (0.0047)	-0.016 (0.0048)	- -	-0.014 (0.0045)
$\Delta rna_t + \Delta rna_{t-1}$	- -	- -	-0.01 (0.0029)	
$\Delta_2 rl_t$	-0.043 (0.0074)	-0.035 (0.0079)	- -	- -
$\Delta_2 rl_{t-3}$	-0.039 (0.0083)	-0.037 (0.0085)	- -	- -
$(D1 + D3)_t$	0.034 (0.0053)	0.035 (0.0053)	0.036 (0.0051)	0.035 (0.0049)
$D4_t * \Delta rna_t$	0.071 (0.025)	0.082 (0.026)	0.093 (0.025)	0.075 (0.025)
DC_t	0.064 (0.0068)	0.062 (0.0069)	0.070 (0.0075)	0.072 (0.0072)
$DCRISIS_t$	0.027 (0.0049)	0.030 (0.0052)	0.028 (0.0051)	0.031 (0.0049)
$DCOVID_{t-2}$	0.11 (0.022)	0.11 (0.023)	0.11 (0.025)	0.14 (0.026)
$\hat{u}_{Rt-1}$	- -	-0.16 (0.020)	0.090 (0.047)	0.097 (0.049)
Constant	0.008 (0.002)	- -	- -	- -
Nonlinear components				
$(\hat{u}_{Rt-1} - 0.2)\hat{u}_{Rt-1}{}^2$	-1.47 (0.19)	- -	- -	-0.95 (0.27)
$f(*) * \Delta(m-p)_{t-1}$	- -	- -	-0.43 (0.11)	-0.37 (0.12)
$f(*) * \Delta^2 p_t$	- -	0.14 (0.091)	- -	- -
$f(*) * \Delta p_{t-1}$	- -	- -	-0.30 (0.043)	-0.29 (0.043)
$f(*) * \Delta_2 rl_t$	- -	- -	-0.074 (0.0099)	-0.069 (0.0097)
$f(*) * \Delta_2 rl_{t-3}$	- -	- -	-0.028 (0.013)	- -
$f(*) * \hat{u}_{Rt-1}$	- -	-0.13 (0.23)	-0.19 (0.053)	-0.15 (0.057)
$f(*) * Constant$	- -	0.069 (0.021)	0.017 (0.0028)	0.013 (0.0028)
T	146	146	146	146
R^2	0.87	0.88	0.88	0.89
$100 * \hat{\sigma}$	1.63%	1.61%	1.57%	1.53%
BIC	-5.02	-4.93	-4.95	-5.01
Nº of parameters	14	18	19	19
Misspecification test	***P-value***	***P-value***	***P-value***	***P-value***
Autocorrelation (2)	0.62	0.95	0.77	0.65
RESET (1)	0.66	-	-	
ARCH (1)	0.52	0.37	0.42	0.59
Huber-White	0.55	0.57	0.60	0.68
Normality	0.96	0.86	0.52	0.52

Notes: Each column of the short-run equations presents coefficients obtained from separate regressions (estimated by OLS for Model A, and by NLS for the rest), with standard errors in parentheses. All reported statistical tests are expressed as p-values. All models used two-step cointegrating residuals estimated by FM-OLS: $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.26 + 8.12\ RNA_{t-1}$. Below are the details of the models:

i) Selected NEC - LSTR(1) with transition function $f(\hat{u}_{Rt-1}) = \{1 + \exp[\gamma\ (\hat{u}_{Rt-1} - c)/\hat{\sigma}_{\hat{u}_{Rt-1}}]\}^{-1}$, estimated model parameters by NLS are $\gamma = -4.11$ (1.49) and $c = 0.11$ (0.021).

ii) Selected LSTR(2) with transition function $f(\Delta y_t) = \{1 - \exp[\gamma\ (\Delta y_t - c_1)((\Delta y_t - c_2)/\hat{\sigma}^2_{\Delta y_t}]\}^{-1}$, estimated model parameters by NLS are $\gamma = -1.99$ (0.80), $c_1 = 0.015$ (0.004) and $c_2 = -0.051$ (0.012) for linear error-correction case (Model C) and estimated model parameters by NLS are $\gamma = -1.86$ (0.80), $c_1 = 0.012$ (0.0057) and $c_2 = -0.046$ (0.014) for NEC case (Model D).

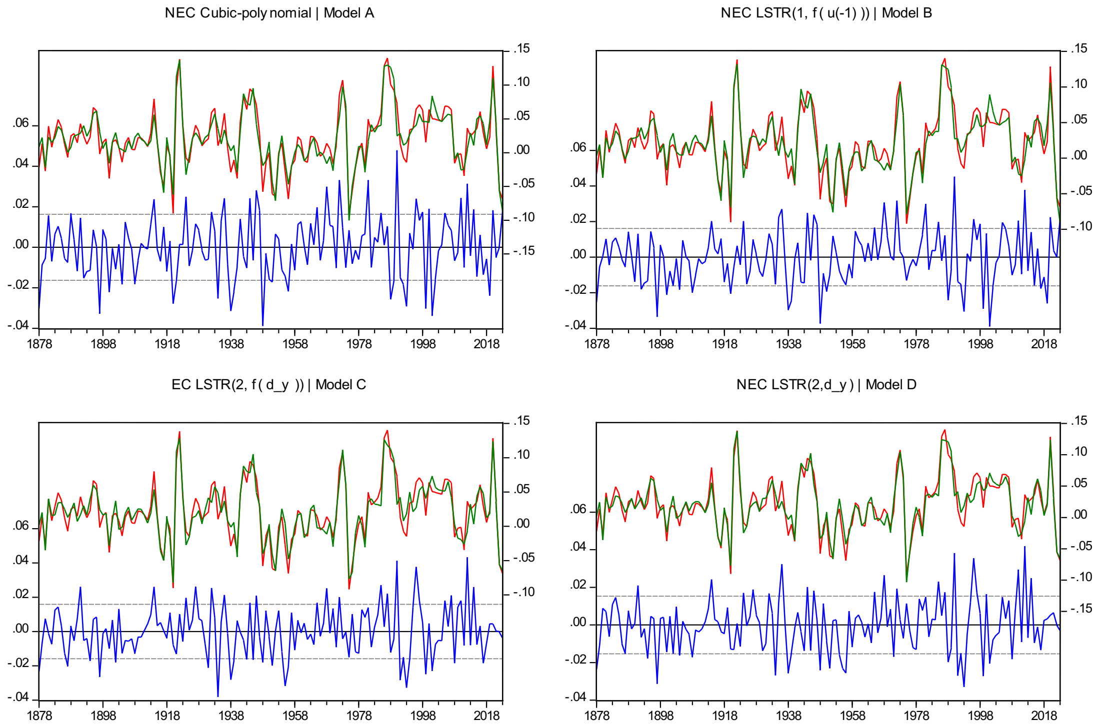

Figure 5. UK Estimated Selected Nonlinear Money Demand Models (1878 – 2023)

Notes: Figure plots the actual values $\Delta(m-p)_t$, fitted values $\Delta\widehat{(m-p)}_t$ and residuals $\hat{\varepsilon}_{vt}$ of selected nonlinear money demand models presented in Table 17. Source: Authors' calculations.

In particular, parameter constancy, which has been a critical and persistent point in money demand models not only from a statistical perspective but also for economic policy evaluation, is evaluated for the *NEC – Cubic polynomial*, Model A using recursive least squares (RLS), as shown in Figure B.2. This stability test can be interpreted as an encompassing test of the additional information included in the forecast or extended period. The results demonstrate that parameter constancy is remarkably robust throughout the sample period, even during the "missing money" episodes of the 1980s, when many traditional relationships broke down. Recursive estimates for the UK, available since 1980, closely align with those presented in the main tables. While occasional deviations are observed, notably in 1980, 1991, 2000 and 2022, one-step residual diagnostics do not indicate persistent instability. The CUSUM and CUSUM square statistics confirm the cumulative sum of recursive residuals stays within the 5% significance interval. Overall, these findings advocate the constancy and well-specification of the nonlinear equilibrium-correction Model A over more than 140 years of data.

Short-run money demand relationships can be satisfactorily modeled with only a few key variables, provided non-linearities and exogenous money demand shocks are accounted for and opportunity cost of money holdings is measured correctly. In all selected models the primary determinants of real money balances are the lagged level of real money balances themselves, the acceleration of inflation ($\Delta^2 p_t$), the previous period's inflation level (Δp_{t-1}), and opportunity cost measures such as Δrna_t and the long-term interest rate ($\Delta_2 rl_t$). It is noteworthy that income growth (Δy_t) plays no direct role in short-term dynamics[7], serving only indirectly as an indicator of parameter changes captured by the *LSTR(2,* $f(\Delta y_t)$) in Model C and D. The specification of interest rates in logarithmic form is not trivial; it is motivated not only by its theoretical consistency with the Baumol-Tobin model but also by its empirical performance. Using logarithms provides linear growth rate approximations that exhibit lower volatility compared to first differences in levels, making them better suited to capturing short-term fluctuations in money demand, as illustrated in Figure B.3.[8]

Following the extended models[9], we avoid using the dummy variable described by Friedman and Schwartz (1982) as a "liquidity preference shift", but we retain the others. The variable $(D1 + D3)_t$ serves as a dummy for both World Wars (WWI: 1914–1918 and WWII: 1939–1945), $D4_t$ accounts for the introduction of Competition and Credit Control during 1971–1975, and DC_t represents the deregulation episodes spanning 1971–1975 and 1986–1989. Notably, in this specification, the $D4_t$ now multiplies the growth rate of Δrna for economic consistency with the opportunity cost measure specified in all models. The dummies for $DCOVID_{t-2}$ and $DCRISIS_t$, previously explained as periods of excess money demand, are also retained. More importantly, the identification of these exogenous episodes of excess money demand—unexplained by the variables within each model—can serve as indicators of inflationary episodes, as demonstrated later. This feature can be captured effectively by any of the models presented previously.

The short-run relationship between money demand and inflation (Δp_t) is consistent with economic theory. As the literature points out, the growth of real balances may decrease as inflationary pressures shape the actual and expected demand for money

[7] Income growth (Δy_{t-k}) and its lags up to four periods (k = 0, 1, 2, 3, 4) were found to be statistically insignificant across all selected models of Table 17, thus rejecting the common factor (COMFAC) restrictions associated with the long-run unitary income elasticity.

[8] Notably, these specifications can be adapted to a negative interest rate framework if such a scenario arises in the data, as recently demonstrated by Escribano and Rodríguez (2023).

[9] See Attfield, Demery and Duck (1995), Friedman and Schwartz (1982), Hendry and Ericsson (1991), Hendry and Mizon (1978), and Lubrano, Pierse and Richard (1986) for details on the introduction of these dummy variables ($D1_t$, $D3_t$, $D4_t$ and DC_t).

balances, thus reducing the real return on physical assets (Friedman and Schwartz 1982). Theoretically the associated inflation's cost tends to increase with the level of inflation, reinforcing the negative effect of inflation at high levels on the quantity of money demanded. However, we do not find empirical evidence of this kind of behavior in the UK demand for real money balances[10].

If the holding of money is viewed as part of a portfolio decision process, the optimal level of cash balance may be determined not by only by single interest rates but by the entire term structure (Heller and Khan 1979; Brand and Cassola 2004). The estimated coefficients and signs on the growth of the short-term opportunity cost measure (Δrna_t) and the long-term interest rates ($\Delta_2 rl_t$) corroborate this kind of behavior, which is well described by the time stucture theory of, aligning with Friedman (1977)'s term structure theory. According to this theory, changes in the term structure affect not only the current opportunity cost of holding money but also agents' expectations of future interest rates. An increase in long-term interest rates, while short-term interest rates remain unchanged, may exacerbate the upward trend in the liquidity ratio by encouraging substitution from long-term to short-term securities, thus inducing a downward effect on the demand for money.

Modeling nonlinearities in the money demand function is crucial. The estimated nonlinear functional forms employed in the selected models are illustrated in Figure 6: (a) the *cubic-polynomial error-correction* of Model A, (b) the *LSTR(1,* $f(\hat{u}_{Rt-1})$*)* transition function of Model B, (c) the *LSTR(2,* $f(\Delta y_t)$) transition function of Model C and D and (d) the *parameter-changing cubic-polynomial error-correction* of Model D. The cubic-polynomial specification in Model A, denoted as $POL\,(u_{rt}) = -1.47\,(\hat{u}_{Rt-1} - 0.2)\hat{u}_{Rt-1}{}^2$, provides a flexible parametric form to capture multiple equilibrium error-correction representations. The linearity test of the error-correction term[11] of Model A also supports a nonlinear error-correction (NEC) representation with asymmetric adjustment towards multiple equilibria (see linearity test of) with a similar dynamic to that proposed by Escribano (1985, 1986, 2004). In Figure 6, depicts this

[10] To test the existence of this non-linear effect we introduce the square of the inflation rate up to 2 lags ($i.e.\ \beta_1 \Delta p_t^2 + \beta_2 \Delta p_{t-1}^2 + \beta_3 \Delta p_{t-2}^2$) in NEC Model A and test the null hypothesis H0: $\beta_1 = \beta_2 = \beta_3 = 0$, which is failed to be rejected at conventional levels of significance (F-stat(3,129)=0.048).

[11] The LM linearity test $\hat{u}_{Rt-1}$ contrast the null hypothesis of linearity in the error-correction term against the alternative of a *cubic-polynomial error-correction* model.

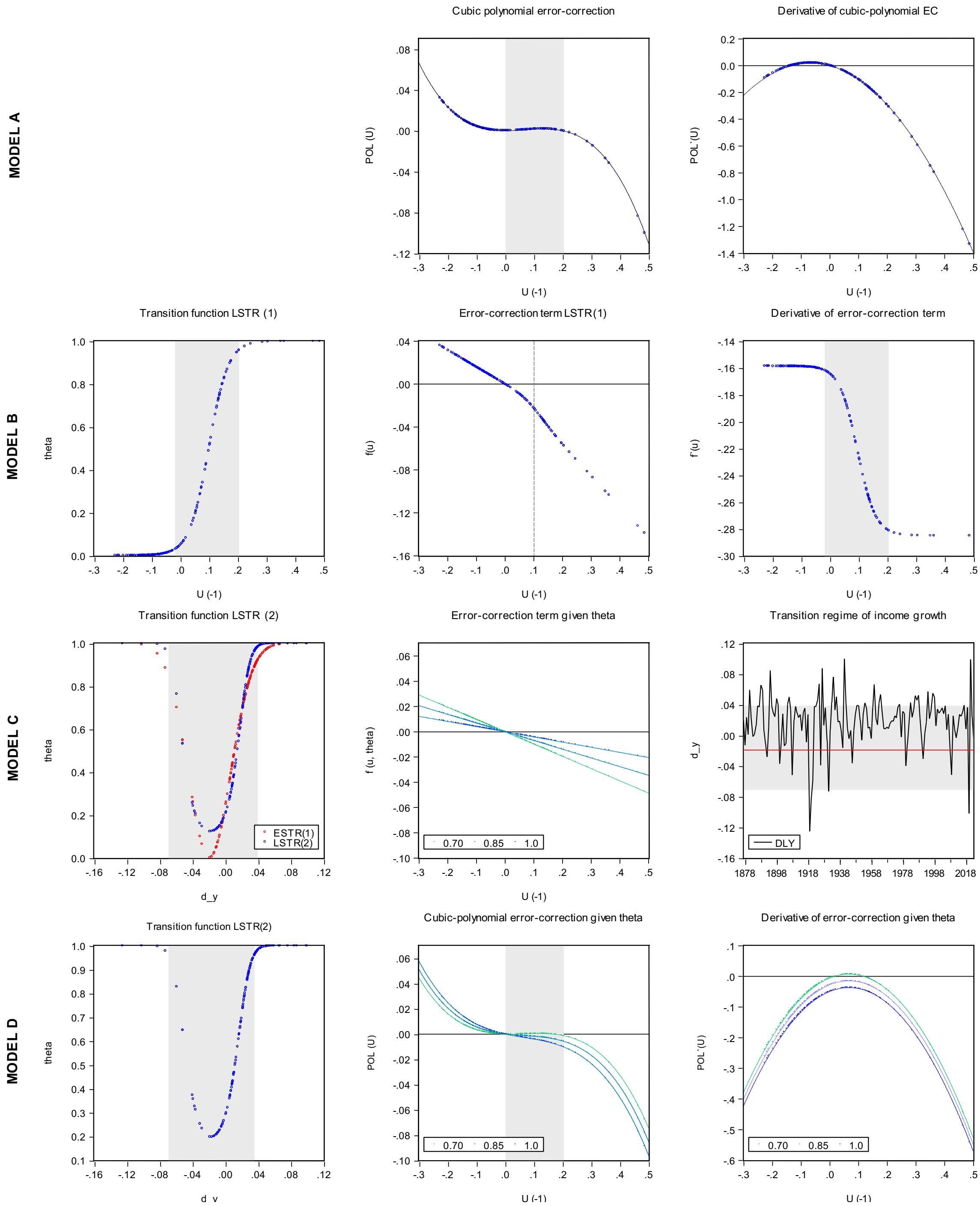

Figure 6. Nonlinear Components of Selected Money Demand Models

Notes: Nonlinear components of each selected model in Table 17 are presented: (a) the cubic-polynomial error-correction $POL(\hat{u}_{rt-1})$ with the equilibrium range (i.e. $POL(U) = 0$) shaded and its first-order derivative; (b) the LSTR(1, $\hat{u}_{Rt-1}$) transition function (θ), the estimated nonlinear error-correction term *f(u)* and its first derivative; (c) the LSTR(2, Δy_t) transition function (θ) and its equivalent ESTR(1, Δy_t), the estimated error-correction term *f(u, θ)* given selected θ values and the income growth (Δy_t) transition regime, with the red line indicating the transition function's minimum value; (d) the LSTR(2, Δy_t) transition function (θ), the cubic-polynomial error-correction term $POL(\hat{u}_{rt-1}, \theta)$ and its first derivative given selected θ values. For all STR models, transition regime is shaded. Source: Authors' calculations.

nonlinear dynamics, with the horizontal axis showing long-run cointegrated residuals $\hat{u}_{Rt-1} = (m - p - y)_{t-1} + 0.26 + 8.12\, RNA_{t-1}$ and the vertical axis showing the corresponding cubic-polynomial function. The nonlinear structure suggests that deviations farther from the equilibrium range undergo larger corrections, consistent with the theoretical buffer-stock (Gandolfi and Lothian 1976; Cuthbertson and Taylor 1987) and target-bounds models (Miller and Orr 1966; Akerlof 1979; Mishkin 1995). Threshold parameters for the equilibrium ranges ($\tau_1 = 0$ and $\tau_2 = 0.2$) remain consistent with findings in Ericsson, Hendry, and Prestwich (1998) and Escribano (2004), reinforcing the robustness of the estimated nonlinearity.

As suggested by Teräsvirta and Eliasson (2001), this nonlinear dynamics can be approximated using a logistic smooth transition regression model, (*LSTR(1,* $f(\hat{u}_{Rt-1})$)) as in Model B. The logistic transition occurs within a range closely resembling the equilibrium range of the cubic polynomial in Model A (see Figure 6), which is approximately between $\tau_1 = -0.05$ and $\tau_2 = 0.24$. In the transition from one regime to another marks the impacts of inflation acceleration and the error-correction term. The adjustment to equilibrium is faster (-0.29) the impact of inflation acceleration diminishes to -0.53 when deviations are above equilibrium range ($\hat{u}_{Rt-1} > 0.24$, i.e., θ = 1) occur, and. Conversely, when deviations fall below equilibrium range ($\hat{u}_{Rt-1} > -0.05$, i.e., θ = 1), inflation acceleration has a greater impact (-0.67), while the effect of the error correction term decreases to -0.19. This dynamic is noteworthy, as deviations from equilibrium also play a significant role in determining the growth of prices, as will be shown later.

Nonlinear dynamics can also be modeled using a time-varying parameter framework, such as in Model C and D, *LSTR(2,* $f(\Delta y_t)$), where income growth indirectly drives parameter changes. In these specifications, the transition occurs smoothly within income growth rates of approximately [-7%, +4%], which align with typical values observed in the sample (see Figure 6).

The LSTR(2) transition function can alternatively be expressed as a simpler first-order exponential function, ESTR(1), which requires fewer parameters for estimation (e.g., only one threshold), as shown in Figure 6 for Model C. The primary distinction is that LSTR(2) estimates inflection points (i.e., where the second derivative equals zero), whereas ESTR(1) identifies critical minimum points (where the first derivative equals zero), of the transition function $\theta = f(*)$. For instance, the estimated value $c_1 = -1.7\%$ indicates the income growth level at which the transition function reaches its minimum. Despite these differences, both specifications yield comparable results, with LSTR(2) exhibiting slightly superior fit to the data. For this reason, LSTR(2) was chosen as the transition function for Model C and D.

Finally for Model D, what stands out is its ability to adjust the cubic nonlinear equilibrium across the range of values where the equilibrium is defined—that is, when the cubic polynomial equals zero (POL (U)=0) — depending on income growth level. This adjustment occurs smoothly, as illustrated in Figure 6 for selected values of the transition function (θ): $\theta = 0.7$ corresponds to Δy_t values between approximately -5.8% and 1.9%, $\theta = 0.85$ corresponds to Δy_t values between approximately -6.3% and 2.6%., and $\theta = 1$ occurs when Δy_t is below -7.8% or above 4.5%. As income growth deviates further from the transition function's minimum ($c_1 \approx -1.6\%$), the cubic polynomial evolves from a multiple equilibrium to a single equilibrium model at $\hat{u}_{Rt} = 0$ when θ=1.

In the case of the estimated STR models, the inclusion of non-constant parameters that vary with the transition variable can complicate the economic interpretation of the model. For Model C and D specifically, the coefficients of the error-correction term ($\hat{u}_{Rt-1}$) and past inflation (Δp_{t-1}) become statistically insignificant near the $c_1 = -1.7\%$, and the limited number of observations at the extremes of the transition range prevents proper evaluation of the coefficients in these regions. Given these considerations, simpler representations as the NEC Model A emerges may be a more parsimonious alternative for approximating nonlinearities without complicating the interpretation of the parameters.

5.6. Searching for Model Encompassing and Out-of-Sample Forecasting

A final step of the model's evaluation is to assess the predictive power of the estimated error-correction models presented in this paper and make a stress test (see Table 17). Since no statistical test in-sample will manifest predictive failure post-sample, forecast evaluation is performed on the basis of their forecast performance out of sample. Hence, three selected and new models have been re-estimated from 1878 to 2002 and results of the out-of-sample dynamic forecasting of the growth of real money balances for the last 20 observations of the sample (2003 – 2023) are plotted for each model in Figure 7.

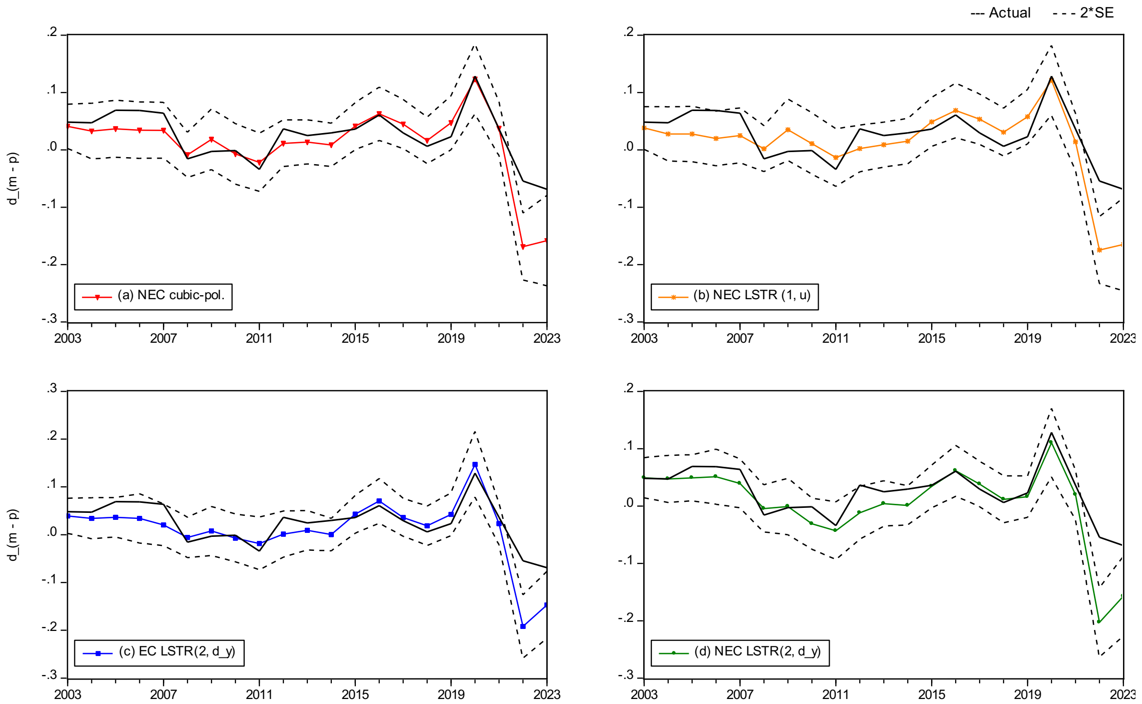

Figure 7. UK Error-Correction Models Out-of-Sample Forecast (2003 - 2023)

Notes: Figure plot the results of the out-of-sample dynamic forecasting of the growth of real money balances $\Delta(m-p)_t$ of the last 20 observations of the sample (2003 - 2023) for the models presented in Table 17. Two forecast standard errors plus or minus the forecasted values are considered to construct the confidence interval. Source: Authors' calculations.

Table 18. UK Out-of-Sample Forecast Evaluation of the last 20 years (2003 – 2023)

Forecast results	*(a) NEC model*	*(b) LSTR(1, $f(\hat{u}_{Rt})$)*	*(c) LSTR(2, $f(\Delta y_t)$)*	*(d) LSTR(2, $f(\Delta y_t)$)*
RMSE	0.036	0.042	0.040	0.042
MAE	0.023	0.031	0.027	0.024
MAPE	112.5	190.8	103.6	167.2
Theil U1	0.31	0.36	0.34	0.35
Theil U2	0.43	0.69	0.51	0.5
Diebold-Mariano test (HLN adjusted)	(a) vs. (b) = 0.00	(b) vs. (c) = 0.57	(a) vs. (c) = 0.34	(a) vs. (d) = 0.33
	-	(b) vs. (d) = 0.97	-	(c) vs. (d) = 0.47
Combination test (2003 – 2023)	0.000	0.006	0.000	0.000
Combination test (1878 – 2023)	0.003	0.000	0.602	0.466

Notes: Each column reports the estimated measures of forecast accuracy: Root Mean Square Error (RMSE), Mean Absolute Error (MAE), Symmetric Mean Absolute Percentage Error (SMAPE), Theil Bounded and Unbounded Inequality Coefficients (Theil U1 and U2) of a dynamic out-of-sample forecast (2001 - 2023) for each of the considered nonlinear error-correction models. The Diebold-Mariano test (Diebold and Mariano 1995) is presented as the p-value of its corresponding two-tailed test, evaluating whether both forecasts have the same accuracy based on RMSE terms. The combination test (Chong and Hendry 1986; Timmermann 2006) test the null hypothesis that forecast i includes all information contained in others and is presented as the p-value for the out-of-sample forecast and full sample case.

All four models show out-of-sample forecasts closely aligned with actual money levels over the forecasted sample. Forecast errors, measured by RMSE and MAE, exhibit similar values across models, with NEC Model A performing best in absolute terms (see Table 18). However, Diebold-Mariano tests show a statistically significant improvement only relative to Model B. Additionally, the Combination Test rejects the null hypothesis for all models[12], suggesting that no single forecast fully encompasses the information provided by others. When considering in-sample dynamic forecasts for the full sample (1878 – 2023), the forecast encompassing test suggests that the LSTR(2) models, C and D, incorporate the forecast information of the others at 10% level of significance. Among these, NEC LSTR(2), Model D, achieves the lowest RMSE (0.015) in absolute terms. However, the predictive gains of these models are not statistically significant at the 5% level, as indicated by the Diebold-Mariano test. This result is consistent with the observational equivalence of all models in terms of predictive power.

Table 19. Parsimonious NEC Money Demand Model of the UK (1878 – 2023): *Cubic-polynomial equilibrium -correction, Model A*

$$\Delta(m-p)_t = \underset{(0.049)}{0.49}\,\Delta(m-p)_{t-1} - \underset{(0.041)}{0.17}\,\Delta(m-p)_{t-2} - \underset{(0.042)}{0.58}\,\Delta^2 p_t - \underset{(0.034)}{0.16}\,\Delta p_{t-1} - \underset{(0.0047)}{0.016}\,\Delta rna_t$$

$$-\underset{(0.0074)}{0.043}\,\Delta_2 rl_t - \underset{(0.0083)}{0.039}\,\Delta_2 rl_{t-3} - \underset{(0.19)}{1.47}\,(\hat{u}_{Rt-1} - 0.2)\hat{u}_{Rt-1}{}^2 + \underset{(0.0053)}{0.034}\,(D1 + D3)_t$$

$$+\underset{(0.025)}{0.071}\,(D4_t * \Delta rna_t) + \underset{(0.0068)}{0.064} DC_t + \underset{(0.0049)}{0.027}\,DCRISIS_t + \underset{(0.022)}{0.11} DCOVID_{t-2} + \hat{\varepsilon}_{vt}$$

Observations	146	Schwarz criterion	-5.02
R^2	0.87	$100 * \hat{\sigma}$	1.63%
Misspecification tests	***P-value***	***Misspecification tests***	***P-value***
Autocorrelation (2)	0.62	Normality	0.96
RESET (1)	0.66	Linearity $\hat{u}_{Rt-1}$	0.00
ARCH (1)	0.52	Huber-White	0.55
Cointegration tests	***T-ratio***	***Cointegration tests***	***T-ratio***
EC (t_1^*)	-4.71	NEC (t_1)	-0.11
NEC (t_3^*)	-6.07	NEC (t_3)	-5.52

Notes: Ordinary least squares (OLS) estimates of two-step nonlinear error-correction model, with standard errors in parentheses. $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.26 + 8.11\, RNA_{t-1}$ is the two-step cointegrating residual by estimated FM-OLS for the whole sample (1877 – 2023). The linearity test of $\hat{u}_{Rt-1}$ test the null hypothesis of a general cubic-polynomial EC representation (i.e. $POL\,(U) = \rho_1\, \hat{u}_{Rt-1} + \rho_2\, \hat{u}_{Rt-1}^2 + \rho_3\, \hat{u}_{Rt-1}^3$) against a linear EC representation (H_0: $\rho_2 = \rho_3 = 0$). The 5% and 10% critical values (c.v.) for rejecting the null hypothesis of no cointegration are provided in Table 2, based on c.v. specifically generated for a sample size of T = 150. Source: Authors' calculations.

[12] The Combination Test, or Forecast Encompassing Test, proposed by Chong and Hendry (1986) and refined by Timmermann (2006), test the null hypothesis that a single forecast encapsulates all the information contained in other individual forecasts, i.e. the difference between the true values and the forecasted values from forecast is not related to the forecasts from all other models.

Overall, the out-of-sample forecasting evaluation of all presented nonlinear models (A, B, C and D) meets a broad range of statistical criteria and encompasses a significant proportion of money demand behavior, explaining over 85% of the variability in money demand over more than 140 years of data. These characteristics are essential for accurately characterizing the underlying data generation process and reinforce the notion that measures of the opportunity cost of money and modeling nonlinear behavior are jointly necessary for establishing a well-specified and stable money demand model, leading to the selection of the NEC Model A as the parsimonious money demand model. Based on its ability to capture nonlinear dynamics, the economic consistency and direct interpretation of its coefficients, and its in-sample and out-of-sample performance, Model A emerges as the most robust and parsimonious choice among the selected models.

Table 20. Impact of Constant Excess Money Demand Episodes on Inflation (1877 - 2023)

$$\Delta p_t = \underset{(0.0065)}{0.022} + \underset{(0.017)}{0.065}\,(D1_t + D3_t + DC_t + DCOVID_{t-2}) + \hat{\epsilon}_t$$

R^2 = 0.19 Observations= 147 *(1877 – 2023)* $100 * \hat{\sigma}$ = 4.84%

Notes: Coefficients from OLS regressions, with standard errors in parentheses, for the whole sample period (1877 - 2023) robust to both heteroskedasticity and autocorrelation, HAC standard errors (Newey and West 1987). Source: Authors` calculations.

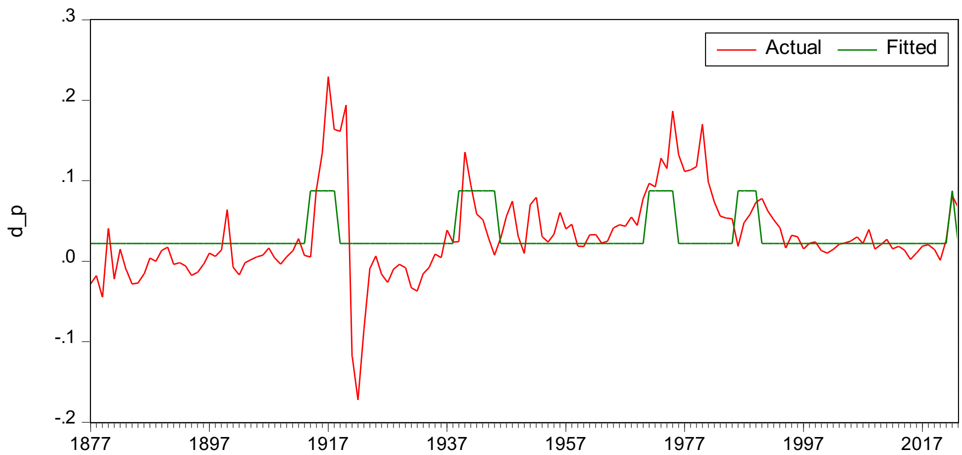

Figure 8. Estimated Contemporaneous Impact of Excess Money Demand Episodes on Long-Run Inflation (1877 - 2023)

Notes: Figure plots the actual values Δp_t and fitted values $\Delta \widehat{p_t}$, of contemporaneous Equation estimated by OLS in Table 20 for the whole sample period (1877 - 2023) robust to both heteroskedasticity and autocorrelation, HAC standard errors (Newey and West 1987). Source: Authors' calculations.

In the presence of excess money demand shocks, due to wars, covid, etc., these excess money holdings over the expected value, anticipate positive constant effects on inflation. Having stable UK long-run money demands, with short-run nonlinear equilibrium corrections, are key elements to identify those periods of excess money demand, generating periods of 6.5% excess inflation over the historical 2.2% average. All coefficients of the dummy variables in Table 20 have equal signs and magnitude. This results (Table 20 and Figure 8) open the door for the use of dynamic money demand equations as an instrument to identify periods of excess money holding that anticipates high inflationary periods.

6. QUANTITATIVE THEORY OF MONEY AND MONETARY POLICY IN THE UK: IMPLICATIONS BASED ON EMPIRICAL EVIDENCE

In the previous section we showed that long-run and short-run UK money demands provide useful information anticipating inflation. Here we show that a) There is no valid common factor (COMFAC) restriction, between the long-run and the short-run money demand parameters and that imposing this COMFAC restriction in a theoretical model might lead us to the wrong conclusions relating the rate of growth of money and inflation and b) controlling for the main long-run economic determinants of real money demand, helps us identifying periods of excess long-run money demand, as well as periods of external excess growth in money demand (wars, regulatory changes, etc.), which are useful predictors of high inflationary periods. Since inflation control is at the end of the day the main goal of central banks, several main monetary policy issues are reevaluated now based on our main UK empirical findings.

Why the UK the rate of growth of real income in the UK has no contemporaneous role in explaining the rate of growth of real balances? For that, we consider a simple model use by Galí (2007), when criticizing the European Central Bank´s (ECB) monetary policy for insisting in the stability of the long-run money demand.

Normalize the target inflation (Δp^X) and trend output growth (Δy^X) to zero, therefore the target money growth is also zero. Now, consider that inflation is proportional to the output deviation from it natural level (y^n), $\Delta p = \lambda(y - y^n)$. Assuming, for simplicity, the existence of a stable long-run money demand independent of the interest rate and given by $m - p = \beta y$. Let the ECB´s *real money gap* be given by $m - p - \beta y^X = \beta y - \beta y^X$.

Now consider that there is a productivity boom with a persistent increase in the natural rate of output ($\Delta y^n > 0$), and that the ECB succeeds in stabilizing inflation ($\Delta p = 0$), by say reducing interest rates to reach an equal increase in output ($\Delta y^n = \Delta y$). The issue now is to evaluate whether money growth is still at the reference value equal to zero, $\Delta m = 0$. For that, Galí (2007) takes first differences of, the long-run money demand equation, $\Delta\, m = \Delta\, p + \beta \Delta\, y$, to say that the money will experience a persistent deviation from its reference value equal to 0. However, this argument is valid only if the common factor (COMFAC) restriction is imposed, but this restriction is generally not supported empirically, as we show later with UK data. Furthermore, in the empirical section we obtain that, the short-run money demand is determined by inflation and by other variables (like long term interest rates, etc.) but the growth of real output is not. Therefore, is not true that there will always be a deviation from the reference value of money growth if the money demand is taken into consideration in a monetary policy targeting inflation.

Furthermore, Ireland (2004) considers the following stochastic money demand equation,

$$m_t - p_t = \beta_y y_t - \beta_{sr} SR_t + e_t \qquad (16)$$
$$e_t = \rho_0 + \rho_1 e_{t-1} + \eta_t$$

where m_t, p_t, y_t and SR_t are I(1) and e_t is a stochastic I(0) variable, like a stationary AR(1), and η_t i.i.d . However, we showed η_t not i.i.d and is not exogenous to the rates of growth of real income and interest rates and in fact it includes those I(0) factors. Equation (16) is useful to explain why in the UK real income affects the money demand in the long run, but its rate of growth does not affect the rate of growth of real money demand. By taking first differences in the first equation of (16), we get (17)

$$\Delta\, \mathrm{m}_t - \Delta\, \mathrm{p}_t = \beta_y \Delta\, \mathrm{y}_t - \beta_{RS} \Delta\, \mathrm{RS}_t + \Delta\, e_t \qquad (17)$$
$$\Delta\, e_t = \rho_0 + (\, \rho_1 - 1)\, e_{t-1} + \eta_t$$

and we are imposing in the first equation of (17) the invalid COMFAC restriction, that equations with variables in levels (16) and with variables differences (17) have the same coefficient values. Once again, without considering the second equation of (17), we get the wrong impression that the rate of growth of real income and the first difference of the nominal short-term interest rates are relevant drivers of the rate of growth of real money demand, based on the QTM. But, in the UK the stochastic term of (17) is not i.i.d, since it is affected by variables like, Δp_t, Δy_t, ΔSR_t, and Δx_t. That is $\eta_t = -\beta_y \Delta y_t + \beta_{RS} \Delta RS_t + \beta_x \Delta x_t + \varepsilon_t$ and therefore (17) is reduced to (18). We show that in the UK the vector Δx_t

includes other significant variables like the rates of growth of short-term and long-term interest rates, etc.

$$\Delta m_t - \Delta p_t = (\rho - 1)\, e_{t-1} + \beta'_x \Delta x_t + \varepsilon_t \tag{18}$$

Therefore, applying the QTM to the UK, neither the rate of growth of real income nor the first difference of the nominal short-term interest rate, are significant drivers of the rate of growth of the real money demand. In fact, the nominal rate of growth of money demand is determined only mainly by inflation (Δp_t), by the equilibrium correction term with one lag (e_{t-1}), and by the rates of growth of short-term and long-term interest rates, all included in the vector Δx_t of (18).

To evaluate the impact of imposing the common factor restriction (COMFAC) in the UK between, the short-run coefficients and the long-run cointegrating coefficients in Figure 9 see the impact of this restriction.

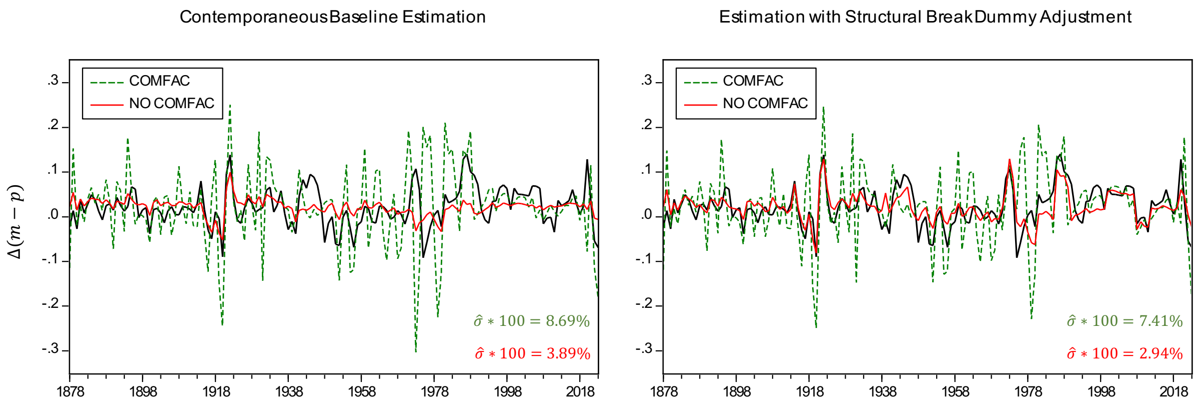

Figure 9. Effects of COMFAC Restrictions on Real Money Demand, $\Delta(m-p)_t$.

Note: COMFAC imposes long-run money demand coefficients, estimated in Equation (11.2) onto the short-run model. In contrast, NO COMFAC specification estimates contemporaneous coefficients (Δp, Δy_t ΔRNA_t) for the period 1877–2023 using OLS robust to both heteroskedasticity and autocorrelation, HAC standard errors (Newey and West 1987). In the second graph we present the results of both estimations after adjusting for structural break dummies ($(D1 + D3)_t$, $D4_t$*Δrna_t, DC_t, $DCRISIS_t$, $DCOVID_{t-2}$). Source: Authors' calculations.

Notice, that by comparing the predicted real balances given by the green line (imposing the COMFAC restriction) with the predictions with the red line (NO-COMFAC restriction imposed), and the actual data (black line), it is clear that by not imposing the COMFAC restriction we can reduce by a large amount of *"artificial excess variability"* generated in the rates of growth of real money demand, $\Delta(m-p)_t$, because it is not in the actual data (black line in Figure 9).

Looking for theoretical models that address the money demand decisions at higher frequencies (short-run money demand), we found the work of Alvarez and Lippi (2014) and Belongia and Ireland (2019) but they do not provide specific empirical relations integrating the short run with the long run in all the equations, like in the Taylor´s rule. This integration comes from Granger´s representation theorem ((Engle and Granger 1987; Johansen 1992). The nonlinearity in money demand functions in the short run can be drawn from Miller and Orr (1966), Akerlof (1979) and Mishkin (1995)´s target-bound model, in which a representative agent's decision to hold money depends on a target level of desired money balances, as well as lower and upper limits that should not be exceeded. Similarly, the buffer-stock models, developed by Gandolfi and Lothian (1976) and Cuthbertson and Taylor (1987), recognize that, if transaction costs are nonzero, it may be optimal for agents to adjust only for relatively large deviations from its long-run equilibrium. Thus, these models imply that the speed of the equilibrium adjustments in money demand functions are likely be smooth and nonlinear, as opposed to abrupt change as in threshold autoregressive models, (TAR).

The traditional view of the QTM states, by solving for inflation in the first equation of (17) to obtain (19), that inflation occurs together changes in the rates of monetary aggregates adjusted for the rate of growth in output and velocity (or short-term interest rates) is not valid for the UK,

$$\Delta P_t = \Delta m_t - \beta_y \Delta y_t + \beta_{RS} \Delta RS_t - \Delta e_t \tag{19}$$

However, as we have seen before, Δe_t in (19) is not i.i.d, nor exogenous, since it is influenced by variables economic that move in opposite directions to real income and short-term interest rates. Then, unless we control in (19) to have exogenous variables and to get residuals that are i.i.d, the conclusions are spurious. Any change in real income, or in interest rates will affect velocity as well and cannot be assumed to be constant. Estimating by OLS, or by HAC, the equations of the QTM, with variables in levels and variables in differences, (27), we get very different empirical coefficient values,

$$\begin{aligned} m_t &= 0.99\, p_t + 1.05 y_t - 0.07 RS_t + \hat{e}_{1t} \\ \Delta m_t &= 0.93 \Delta p_t + 0.46 \Delta y_t - 0.009 \Delta RS_t + \Delta \hat{e}_{2t} \end{aligned} \tag{27}$$

The reason is because the residuals from both equations are neither white noise nor the explanatory variables are exogenous. However, in the first equation this is not relevant because the cointegration estimator is super-consistent even in the presence of endogenous regressors, but this is not true when we have stationary variables (in first differences) as in the second equation of (27).

One way to orthogonalized the explanatory variables of the cointegrating equation of the QTM and getting white noise residuals, is to run a linear ARDL of the residuals of the cointegrating equation as in (28), conditioning on all the I(0) variables entering in the QTM equation,

$$\begin{aligned}\hat{e}_{1t} = &-0.89 + 1.8\,\hat{e}_{1t-1} - 1.15\,\hat{e}_{1t-2} + 0,29\,\hat{e}_{1t-3} + \\ &-0.67\Delta p_t + 0.73\Delta p_{t-1} - 0.17\Delta p_{t-2} + \\ &-1.04\Delta y_t + 0.96\;\Delta y_{t-1} - 0.27\;\Delta y_{t-2} + \\ &+0.06\Delta RS_t + -0.06\Delta RS_{t-1} + 0.02\Delta RS_{t-2} + \\ &-0.003\Delta RS_{t-3} + \hat{\varepsilon}_{1t}\end{aligned} \tag{28}$$

Taking first difference of the cointegrating equation of (27) and from (28) we observe that the contemporaneous rates of growth of real income and of short-term interest rates cancel out in (29),

$$\begin{aligned}\Delta m_t = &\;0.99\;\Delta p_t + 1.05\Delta y_t - 0.07\Delta RS_t + \Delta\hat{e}_{1t} \\ \Delta\hat{e}_{1t} = &-0.89 + 0.8\,\hat{e}_{1t-1} - 1.15\,\hat{e}_{1t-2} + 0,29\,\hat{e}_{1t-3} + \\ &-0.67\Delta p_t + 0.73\Delta p_{t-1} - 0.17\Delta p_{t-2} + \\ &-1.04\Delta y_t + 0.96\;\Delta y_{t-1} - 0.27\;\Delta y_{t-2} + \\ &+0.06\Delta RS_t + -0.06\Delta RS_{t-1} + 0.02\Delta RS_{t-2} + \\ &-0.003\Delta RS_{t-3} + \hat{\varepsilon}_{1t}\end{aligned} \tag{29}$$

A fully specified model of the QTM, with white noise residuals is reduced to equation (30).

$$\begin{aligned}\Delta m_t = &\;0.38\;\Delta p_t + \Delta\hat{e}_{1t} \\ \Delta\hat{e}_{1t} = &-0.89 + 0.8\,\hat{e}_{1t-1} - 1.15\,\hat{e}_{1t-2} + 0,29\,\hat{e}_{1t-3} + \\ &-0.67\Delta p_t + 0.73\Delta p_{t-1} - 0.17\Delta p_{t-2} + \\ &+0.96\;\Delta y_{t-1} - 0.27\;\Delta y_{t-2} + \\ &+ -0.06\Delta RS_{t-1} + 0.02\Delta RS_{t-2} + \\ &-0.003\Delta RS_{t-3} + \hat{\varepsilon}_{1t} = \\ &= \Delta\hat{e}_{f,1t-1} + \hat{\varepsilon}_{1t}\end{aligned} \tag{30}$$

This system of equations (30) has important implications to forecast inflation and to determine the rate of growth of money supply consistent with an inflationary target, opening the door of the use money as an instrument in monetary policy targeting inflation.

Decomposing the system (30) into the part that is forecastable at t-1 and the part that determined by contemporaneous variables, the system (30) is reduced to equation (31),

$$\Delta m_t = 0.38 \ \Delta p_t + \Delta \hat{e}_{f,1t-1} + \hat{\varepsilon}_{1t} \tag{31}$$

Notice that the elements of velocity of circulation of money now, $\Delta \hat{e}_{f,1t-1} + \hat{\varepsilon}_{1t}$, do not depend neither on contemporaneous values of the inflation rate, on the rate of real income growth, nor on changes in short-term interest rates. Therefore, in (31) we have a monetary policy equation for inflation, consistent with a fully specified QTM equations base exogenous variables and white noise residuals, or

$$\Delta p_t = 3.2 \ (\Delta \mathrm{m}_t - \Delta \hat{e}_{f,1t-1}) - \hat{\varepsilon}_{1t} \tag{32}$$

In percentage terms equation (32) establish that the next period rate of inflation can be controlled by fixing the rate of growth of money supply in the hand of the central banks, net of the predicted rate of change in velocity of circulation of money at time t-1, based on stable real money demand estimates. That is,

$$100\Delta p_{t+1} = 3.2 \ (100\Delta \mathrm{m}_{t+1} - 100\Delta \hat{e}_{f,1t}) \tag{33}$$

Say that the target inflation is equal to long-run historical average of 2.2%, previously estimated in Table 20. Then from (31), the corresponding rate of growth of money, net of the last period predicted rate of change on velocity of circulation of money demand, should be set equal to 0.84%,

$$100(\Delta \mathrm{m}_{t+1} - \Delta \hat{e}_{f,1t}) = 0.38 \ x \ 2.2\% = 0.84\% \tag{34}$$

Therefore, monetary policy rule (34), stablish that the rate of growth of real income, the rate of change of short-term interest rates and the equilibrium correction terms provide useful information to anticipate at time t-1 the changes the velocity of circulation of the money $\Delta \hat{e}_{f,1t}$. However, their contemporaneous values at time t+1 are of no use to calculate the next period rate for growth of net money supply, Δm_{t+1}, that could be fixed at the rate of growth of 0.84%, consistent with the target inflation of 2.2%.

Now, we are ready to answer the question of, what concept of velocity o circulation of money allows us to do monetary policy analysis based on the QTM equation? The answer is the “net nominal money demand (me_t)” obtained from the money demand equation (30), or by the one obtained with the more money demand equation given in the nonlinear equilibrium correction (NEC) model of Table 7.

Subtracting the fitted values of (28), say $\hat{e}_{f,t}$, from m_t we get what we call "net nominal money demand" that will be very useful, $me_t = m_t - \hat{e}_{f,t}$. Using this net nominal money demand variable in the QTM cointegrating equation, the explanatory variables are exogenous by construction and the residual are white noise. Therefore, now there should be no relevant difference in the estimated coefficients of QTM equation, if we analyze the equation with variables in differences, with variables in log-levels or with variables in deviations from steady state (using say the HP filters with $\lambda = 100$), as we obtained estimating now the system (35),

$$\begin{aligned} me_t &= 0.99\, p_t + 1.05 y_t - 0.07 RS_t + \hat{\varepsilon}_{1t} \\ \Delta\, me_t &= 0.99\, \Delta\, p_t + 1.06\, \Delta\, y_t - 0.07\, \Delta\, RS_t - \hat{\varepsilon}_{2t-1} + \hat{\varepsilon}_{2t} \\ \widehat{\mathrm{me}}_t &= 0.90\, \hat{\mathrm{p}}_t + 0.99\, \hat{\mathrm{y}}_t - 0.07\, \widehat{\mathrm{RS}}_t + \hat{\varepsilon}_{3t} \\ \Delta\, \widehat{\mathrm{me}}_t &= 0.90 \Delta\, \hat{\mathrm{p}}_t + 0.99 \Delta\, \hat{\mathrm{y}}_t - 0.07 \Delta\, \widehat{\mathrm{RS}}_t - \hat{\varepsilon}_{4t-1} + \hat{\varepsilon}_{4t} \end{aligned} \quad . \qquad (35)$$

Therefore, to do policy analysis based on the QTM equation we cannot assume that the parameters of the equation do not change (COMFAC restriction), if the variables entering the QTM equation are in levels, or alternatively they are entering in deviations from steady-state (business cycle components), or in first differences (or in rates of growth) or in differences from the steady-state (first differences of business cycle components). However, if the money concept of money entering in QTM is the net nominal money demand, then the parameters of all the equations do not change significantly, allowing us to do policy analysis based on any of the equations of (35). This important issue is usually not considered neither in theoretical macroeconomic models, nor when doing policy analysis based on the QTM. In summary, money matters for monetary policy targeting inflation. But is not the nominal amount of money, is the "net nominal money demand", me_t, obtained from estimating stable money demand equations that generate a "net velocity of circulation of money", equal to ($\hat{\boldsymbol{\varepsilon}}_{\mathbf{1t}}$ *or* $\hat{\boldsymbol{\varepsilon}}_{\mathbf{3t}}$) in (35).

7. CONCLUSIONS

Since the influential work of Friedman and Schwartz (1963, 1982) and later on of Hendry and Ericsson (1991) on the monetary history of the US and the UK from 1876 to 1975, there has been a great concern in the literature about the potential instability of money demand functions. According to the traditional prescription of the Quantity Theory of Money (QTM) expressed in growth rates says that if the velocity of circulation of money is close to being constant ($\Delta v_t \approx 0$), central banks could achieve a zero-inflation rate by setting the growth rate of money supply equal to the growth rate of real income. However, it is well known that velocity, Δv_t, is far from been constant. We also show that is far from having exogenous variables, that satisfies a COMFAC restriction and is not having i.i.d. errors, with important monetary policy implications. Finding a stable relation for -Δv_t is nothing more than finding a stable real money demand equation and would allow us to study under what conditions velocity would close zero, or close to reach a particular inflation target. By means of nonlinear cointegration and nonlinear error-correction models, this paper presents evidence of alternative competing UK money demand models that provide stable long-run and short-run estimates from 1874 to 2023, apart from constant interventions to control for war periods, financial crisis, covid, etc. They are based on broad money measures and an adjusted opportunity cost of holding money, suggested Ericsson et al. (1998). These equations provide key elements to identify periods of excess money demand generating periods of 6.5% excess inflation, over the historical 2.2% average. In summary, money demand estimates provide a simple policy rule for the rate of growth of “net money demand” and provides additional cross-check instruments for monetary policy to reach inflation targets. Finally, stable money demand estimates are useful to identify spurious transmission channels of monetary policy when theoretical models, impose invalid common factor (COMFAC) restrictions and are based on a QTM not fully specified and based on exogenous variables and white noise residuals. The “net concept of velocity of circulation money” introduced here allow us to do policy analysis based on the QTM equations.

REFERENCES

Akerlof, George A. 1979. “The Case against Conservative Macroeconomics: An Inaugural Lecture.” *Economica* 46 (183): 219. https://doi.org/10.2307/2553741.

Alvarez, Fernando, and Francesco Lippi. 2014. “Persistent Liquidity Effects and Long-Run Money Demand.” *American Economic Journal: Macroeconomics* 6 (2): 71–107. https://doi.org/10.1257/MAC.6.2.71.

Arranz, Miguel A., and Alvaro Escribano. 2000. “Cointegration Testing Under Structural Breaks: A Robust Extended Error Correction Model.” *Oxford Bulletin of Economics and Statistics* 62 (1): 23–52. https://doi.org/10.1111/1468-0084.00158.

Attfield, Clifford LF, David Demery, and Nigel W. Duck. 1995. “Estimating the UK Demand for Money Function: A Test of Two Approaches.” *DISCUSSION PAPER-UNIVERSITY OF BRISTOL DEPARTMENT OF ECONOMICS.*

Ball, Laurence. 2012. “Short-Run Money Demand.” *Journal of Monetary Economics* 59 (7): 622–33.

Baumol, William J. 1952. “The Transactions Demand for Cash: An Inventory Theoretic Approach.” *The Quarterly Journal of Economics* 66 (4): 545–56. https://doi.org/10.2307/1882104.

Belongia, Michael T., and Peter N. Ireland. 2019. “The Demand for Divisia Money: Theory and Evidence.” *Journal of Macroeconomics* 61 (September): 103128. https://doi.org/10.1016/J.JMACRO.2019.103128.

Benati, Luca, Robert E. Lucas, Juan Pablo Nicolini, and Warren Weber. 2021. “International Evidence on Long-Run Money Demand.” *Journal of Monetary Economics* 117 (January): 43–63. https://doi.org/10.1016/J.JMONECO.2020.07.003.

Brand, Claus, and Nuno Cassola. 2004. “A Money Demand System for Euro Area M3.” *Applied Economics* 36 (8): 817–38. https://doi.org/10.1080/0003684042000229541.

Canova, Fabio, and Tobias Menz. 2011. “Does Money Matter in Shaping Domestic Business Cycles? An International Investigation on JSTOR.” *Journal of Money, Credit and Banking* 43: 577–607. https://www.jstor.org/stable/20870068?seq=1#metadata_info_tab_contents.

Chong, Yock Y., and David F. Hendry. 1986. “Econometric Evaluation of Linear Macro-Economic Models.” *The Review of Economic Studies* 53 (4): 671–90. https://doi.org/10.2307/2297611.

Cuthbertson, Keith, and Mark P. Taylor. 1987. “The Demand for Money: A Dynamic Rational Expectations Model.” *The Economic Journal* 97: 65. https://doi.org/10.2307/3038230.

Dickey, David A, and Wayne A Fuller. 1979. “Distribution of the Estimators for Autoregressive Time Series With a Unit Root.” *Source: Journal of the American Statistical Association*. Vol. 74.

Diebold, Francis X., and Roberto S. Mariano. 1995. “Comparing Predictive Accuracy.” *Https://Doi.Org/10.1198/073500102753410444* 13: 253–63. https://doi.org/10.1198/073500102753410444.

Engle, Robert F, and C W J Granger. 1987. “Co-Integration and Error Correction: Representation, Estimation, and Testing.” Vol. 55. https://about.jstor.org/terms.

Ericsson, Neil R., David F. Hendry, and Kevin M. Prestwich. 1998. “The Demand for Broad Money in the United Kingdom, 1878–1993.” *Scandinavian Journal of Economics* 100 (1): 289–324. https://doi.org/10.1111/1467-9442.00103.

Escanciano, Juan-Carlos, and Álvaro Escribano. 2009. "Econometrics: Non-Linear Cointegration." *Complex Systems in Finance and Econometrics*, 203–15. https://doi.org/10.1007/978-1-4419-7701-4_11.

Escribano, Álvaro. 1985. "Nonlinear Error-Correction: The Case of Money Demand in the UK (1878-1970)." *University of California, Mimeo*.

———. 1986. "Identification and Modelling of Economic Relationships in a Growing Economy." *University of California, San Diego, Department of Economics*.

———. 1987. "Error-Correction Systems: Nonlinear Adjustments to Linear Long-Run Relationships." *LIDAM Discussion Papers CORE*. Université catholique de Louvain, Center for Operations Research and Econometrics (CORE). https://ideas.repec.org/p/cor/louvco/1987030.html.

———. 2004. "Nonlinear Error Correction: The Case of Money Demand in the United Kingdom (1878-2000)." *Macroeconomic Dynamics* 8 (1): 76–116. https://doi.org/10.1017/S1365100504030019.

Escribano, Álvaro, and Clive Wilian John Granger. 1998. "Investigating the Relationship between Gold and Silver Prices." *Journal of Forecasting* 17: 81–107. https://e-archivo.uc3m.es/handle/10016/2558.

Escribano, Álvaro, and Santiago Mira. 2002. "Nonlinear Error Correction Models." *Journal of Time Series Analysis* 23 (5): 509–22. https://doi.org/10.1111/1467-9892.00276.

Escribano, Álvaro, and Gerard A. Pfann. 1998. "Non-Linear Error Correction, Asymmetric Adjustment and Cointegration." *Economic Modelling* 15 (2): 197–216. https://doi.org/10.1016/S0264-9993(97)00023-0.

Escribano, Álvaro, and Juan Andrés Rodriguez. 2023. "Monetary Trends in the UK and the USA from 1874 to 2020: A Nonlinear Approach to Money Demand," July. https://e-archivo.uc3m.es/handle/10016/37911.

Escribano, Álvaro, M. Teresa Santos, and Ana E. Sipols. 2008. "Testing for Cointegration Using Induced-Order Statistics." *Computational Statistics* 23 (1): 131–51. https://doi.org/10.1007/S00180-007-0081-9/METRICS.

———. 2018. "A New Cramer-Von Misses Cointegration Test with Application to Environmental Kuznets Curve." *Applied Economics* 50 (36): 3966–78. https://doi.org/10.1080/00036846.2018.1430348.

Escribano, Álvaro, Ana E. Sipols, and Felipe Aparicio. 2006a. "Range Unit-Root (RUR) Tests: Robust against Nonlinearities, Error Distributions, Structural Breaks and Outliers." *Journal of Time Series Analysis* 27 (4): 545–76. https://doi.org/10.1111/J.1467-9892.2006.00474.X.

———. 2006b. "Nonlinear Cointegration and Nonlinear Error Correction: Record Counting Cointegration Tests." *Communications in Statistics* 35 (4): 939–56. https://doi.org/10.1080/03610910600880351.

Escribano, Álvaro, and María Torrado. 2018. "Nonlinear and Asymmetric Pricing Behaviour in the Spanish Gasoline Market." *Studies in Nonlinear Dynamics and Econometrics* 22 (5). https://doi.org/10.1515/SNDE-2017-0065/MACHINEREADABLECITATION/RIS.

Escribano, Álvaro, and Dandan Wang. 2021. "Mixed Random Forest, Cointegration, and Forecasting Gasoline Prices." *International Journal of Forecasting* 37 (4): 1442–62. https://doi.org/10.1016/J.IJFORECAST.2020.12.008.

Fisher, Irving. 1911. *The Purchasing Power of Money: Its` Determination and Relation to Credit Interest and Crises*. *Cosimo, Inc*. Republishe. Vol. 74. New York.

https://doi.org/10.2307/2339747.

Friedman, Benjamin M, and Kenneth N Kuttner. 1992. "Money, Income, Prices, and Interest Rates." Vol. 82.

Friedman, Milton. 1956. "The Quantity Theory of Money: A Restatement." *Studies in the Quantity Theory of Money* 5: 3–31. http://localhost:8080/xmlui/handle/123456789/6801.

———. 1961. "The Demand for Money." *American Philosophical Society Proceedings* 105: 259–64.

———. 1977. "Time Perspective in Demand for Money." *The Scandinavian Journal of Economics* 79 (4): 397. https://doi.org/10.2307/3439699.

Friedman, Milton, and Anna J. Schwartz. 1963. *A Monetary History of the United States, 1867–1960*. https://www.nber.org/books-and-chapters/monetary-history-united-states-1867-1960.

———. 1982. *Monetary Trends in the United States and United Kingdom: Their Relation to Income, Prices, and Interest Rates, 1867-1975*. University of Chicago Press. https://www.nber.org/books-and-chapters/monetary-trends-united-states-and-united-kingdom-their-relation-income-prices-and-interest-rates.

Galí, Jordi. 2007. "A Comment on Fischer, Lenza, Pill, and Reichlin's. ECB Experience 1999-2006."

Gandolfi, Arthur E, and James R Lothian. 1976. "The Demand for Money from the Great Depression to the Present." Vol. 66. American Economic Association.

Heller, H. Robert, and Mohsin S. Khan. 1979. "The Demand for Money and the Term Structure of Interest Rates." *Https://Doi.Org/10.1086/260742* 87 (1): 109–29. https://doi.org/10.1086/260742.

Hendry, David F., and Neil R. Ericsson. 1985. "Assertion without Empirical Basis: An Econometric Appraisal of Monetary Trends in ... The United Kingdom." https://www.federalreserve.gov/econres/ifdp/assertion-without-empirical-basis-an-econometric-appraisal-of-monetary-trends-in-the-united-kingdom.htm.

———. 1991. "An Econometric Analysis of U.K. Money Demand in 'Monetary Trends in the United States and the United Kingdom' by Milton Friedman and Anna J. Schwartz." *American Economic Review* 81 (1).

Hendry, David F., and Grayham E. Mizon. 1978. "Serial Correlation as a Convenient Simplification, Not a Nuisance: A Comment on a Study of the Demand for Money by the Bank of England." *The Economic Journal* 88 (351): 549. https://doi.org/10.2307/2232053.

Hwan Seo, Myung. 2011. "Estimation of Nonlinear Error Correction Models." *Econometric Theory* 27 (2): 201–34. https://doi.org/10.1017/S026646661000023X.

Ireland, Peter. 2004. "Money's Role in the Monetary Business Cycle." *Journal of Money, Credit and Banking* 36 (6): 969–83. https://econpapers.repec.org/RePEc:mcb:jmoncb:v:36:y:2004:i:6:p:969-83.

Johansen, Søren. 1992. "Cointegration in Partial Systems and the Efficiency of Single-Equation Analysis." *Journal of Econometrics* 52 (3): 389–402. https://doi.org/10.1016/0304-4076(92)90019-N.

Jong, Robert M. De. 2001. "Nonlinear Estimation Using Estimated Cointegrating Relations." *Journal of Econometrics* 101 (1): 109–22. https://doi.org/10.1016/S0304-4076(00)00075-0.

Kapetanios, George, Yongcheol Shin, and Andy Snell. 2006. "Testing for Cointegration in

Nonlinear Smooth Transition Error Correction Models.” *Econometric Theory* 22 (2): 279–303. https://doi.org/10.1017/S0266466606060129.

Kiliç, Rehim. 2011. “Testing for Co-Integration and Nonlinear Adjustment in a Smooth Transition Error Correction Model.” *Journal of Time Series Analysis* 32 (6): 647–60. https://doi.org/10.1111/J.1467-9892.2011.00722.X.

Laidler, David EW. 1993. “The Demand for Money: Theories, Evidence, and Problems.” *New York: Harper and Collins* 4th editio.

Longbottom, Andrew, and Sean Holly. 1985. “Econometric Methodology and Monetarism: Professor Friedman and Professor Hendry on the Demand for Money.” *London Business School, Centre for Economic Forecasting Discussion Paper*, 1985.

Longbottom, J. A., and S. Holly. 1985. “The Role of Time Series Analysis in the Evaluation of Econometric Models.” *Journal of Forecasting* 4 (1): 75–87. https://doi.org/10.1002/FOR.3980040111.

Lubrano, M., R. G. Pierse, and J. F. Richard. 1986. “Stability of a U.K. Money Demand Equation: A Bayesian Approach to Testing Exogeneity.” *The Review of Economic Studies* 53 (4): 603–34. https://doi.org/10.2307/2297608.

Lucas, Robert E. J.R. 2007. “Central Banking: Is Science Replacing Art? In Monetaery Policy: A Journet from Theory to Practice.” *European Central Bank.*

Miller, Merton H., and Daniel Orr. 1966. “A Model of the Demand for Money by Firms.” *Quarterly Journal of Economics* 80 (3): 413–35. https://doi.org/10.2307/1880728.

Mishkin, Frederic S. 1995. “Symposium on the Monetary Transmission Mechanism.” *Journal of Economic Perspectives* 9 (4): 3–10. https://doi.org/10.1257/jep.9.4.3.

Phillips, Peter C.B., and Bruce E. Hansen. 1990. “Statistical Inference in Instrumental Variables Regression with i(1) Processes.” *Review of Economic Studies* 57 (1): 99–125. https://doi.org/10.2307/2297545.

Qureshi, Irfan A. 2021. “THE ROLE OF MONEY IN FEDERAL RESERVE POLICY.” *Macroeconomic Dynamics* 25 (8): 2037–57. https://doi.org/10.1017/S1365100519000956.

Saikkonen, Pentti. 2005. “Stability Results for Nonlinear Error Correction Models.” *Journal of Econometrics* 127 (1): 69–81. https://doi.org/10.1016/J.JECONOM.2004.03.001.

Selden, Richard. 1956. “Monetary Velocity in the United States.” *Studies in the Quantity Theory of Money* 3.

Teräsvirta, Timo, and Ann Charlotte Eliasson. 2001. “Non-Linear Error Correction and the UK Demand for Broad Money, 1878–1993.” *Journal of Applied Econometrics* 16 (3): 277–88. https://doi.org/10.1002/JAE.611.

Teräsvirta, Timo, Dag Tjostheim, and Clive W. J. Granger. 2010. “Modelling Nonlinear Economic Time Series.” *OUP Catalogue*. https://ideas.repec.org/b/oxp/obooks/9780199587155.html.

Thomas, R, and N Dimsdale. 2017. “A Millennium of UK Data.” Bank of England OBRA Dataset. 2017. http://www.bankofengland.co.uk/research/Pages/onebank/threecenturies.aspx%0A %0A.

Timmermann, Allan. 2006. “Forecast Combinations.” *Handbook of Economic Forecasting* 1 (Chapter 4): 135–96. https://doi.org/10.1016/S1574-0706(05)01004-9.

Tobin, James. 1965. “Money and Economic Growth.” *Econometrica* 33 (4): 671. https://doi.org/10.2307/1910352.

Wagner, Martin, and Seung Hyun Hong. 2016. "Cointegrating Polynomial Regressions: Fully Modified Ols Estimation and Inference." *Econometric Theory* 32 (5): 1289–1315. https://doi.org/10.1017/S0266466615000213.

Wieland, Volker, Guenter Beck, Volker Wieland, and Guenter Beck. 2010. "Money in Monetary Policy Design: Monetary Cross-Checking in the New-Keynesian Model." https://econpapers.repec.org/RePEc:ecb:ecbwps:20101191.

Woodford, Michael. 2003. "Interest and Prices : Foundations of a Theory of Monetary Policy," 804.

———. 2007. "Globalization and Monetary Control," August. https://doi.org/10.3386/W13329.

A. DATA APPENDIX

Variable	Definition	Sources
m_t	Nominal money stock (millions of £) measured as $\ln(M_t)$.	[1]-[3], [5]
p_t	Price level (index 1929=100) measured as $\ln(P_t)$.	[1]-[3], [5]
y_t	Real income (millions of £) measured as $\ln(Y_t)$.	[1]-[3], [6]
$(m-p)_t$	Real money balances (broad measure) measured as ln $(M_t\ /\ P_t)$.	
$(m-p-y)_t$	Inverse of money velocity measured as ln $(M_t\ /\ (Y_t * P_t))$.	
v_t	Money velocity measured as ln $(Y_t * P_t\ /\ M_t) = (y+p-m)_t$.	
H_t	Nominal high-powered money (Millions of £).	[1]-[3], [5]
RS_t	Short-term nominal interest rate (fraction, percent per annum).	[1], [5]
RL_t	Long-term interest rate (fraction, percent per annum).	[4], [5]
rna_t	Opportunity cost measure[13], calculated as $RS_t * ({H_t}^A/{M_t}^A)/0.25$.	
$D1_t$	Dummy that takes 1 for WWI (1914 – 1918) and 0 otherwise.	
$D3_t$	Dummy that takes 1 for WWII (1939 – 1945) and 0 otherwise.	
$D4_t$*Δrna_t	Product of dummy that takes 1 for first period of Financial and Credit Deregulation (1971 – 1975) and 0 otherwise, and first difference of rna_t	
DC_t	Dummy takes 1 for both periods of UK Financial and Credit Deregulation (from 1971-1975 and 1986-1989) and 0 otherwise.	
$DCRISIS_t$	Dummy for the financial crisis. It takes value of 1 from 2000 to 2007, -1 from 2008 to 2011, and 0 otherwise.	
$D9_t$	Dummy that takes 1 for the financial crisis shock (2009) and 0 otherwise	
$DCOVID_t$	Dummy for the global economic recession caused by the COVID-19 pandemic in 2020. It takes value of 1 in 2020 and 0 otherwise.	

[1] Friedman and Schwartz (1982).

[2] Hendry and Ericsson (1991).

[3] Escribano (2004).

[4] Thomas and Dimsdale (2017)

[5] Bank of England (BoE).

[6] Office of National Statistics (ONS).

[7] Organisation for Economic Co-operation and Development (OECD)

[13] As advised by Ericsson, Hendry, and Prestwich (1998), RNA_t is constructed using the actual values of nominal high-powered money (${H_t}^A$) and broad money stock (${M_t}^A$), which denote series without any rescaling in the transition years. For ease of comparison to other interest rate measures, RNA_t is rescaled by the constant $c = 0.25$.

B. APPENDIX

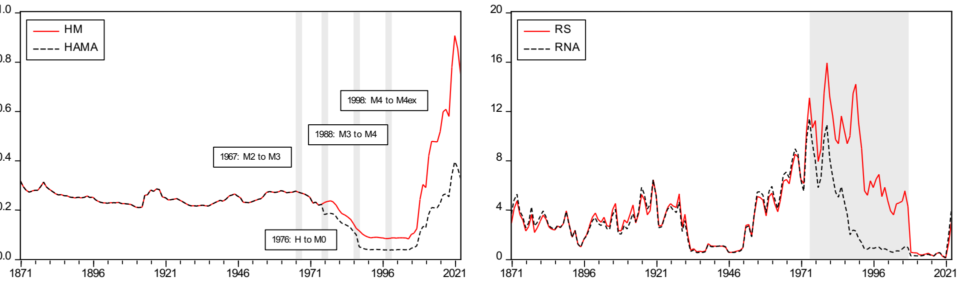

Figure B.1. Historical Comparison Between High-Powered Money to Broad Money Ratios, and Short-Run Interest Rates Measures

Notes: First figure plot the proportion of high-powered money to broad money, using actual ($HAMA = {H_t}^A/{M_t}^A$) and spliced values ($HM = H_t/M_t$) and, second figure plots the short-term opportunity cost measures, RNA_t, and RS_t, for the whole sample period (1871 – 2023) in percent per annum.

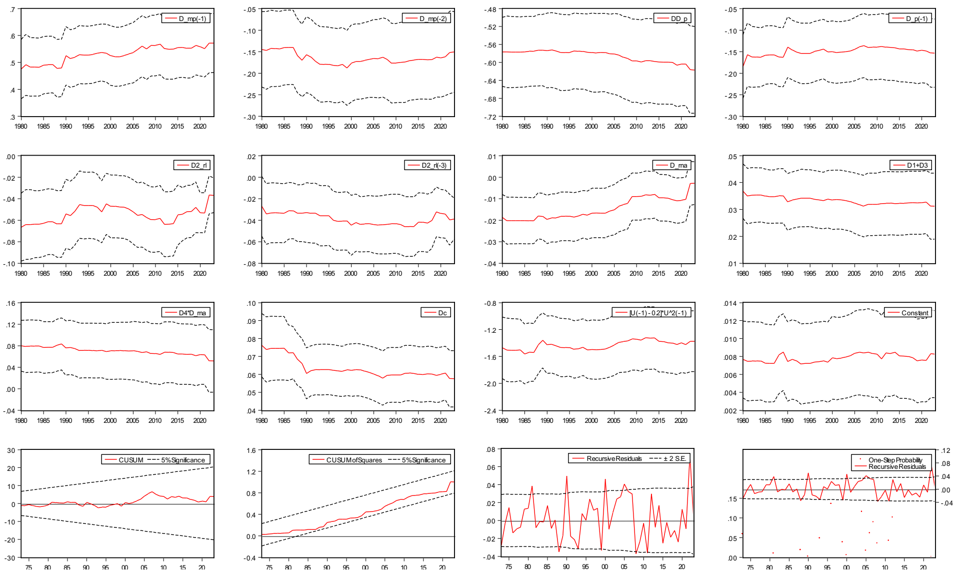

Figure B.2. Selected NEC Model A Constancy Diagnostic: Recursive Estimates

Notes: Figure plots the recursively estimated coefficients, the 1-step residuals plus-or-minus twice their recursively estimated standard errors (dotted black lines), the cumulative sum of the recursive residuals and its sum of squares (CUSUM) together with the 5% critical lines (dotted black lines) and the 1-step forecast probability that parameter constancy would be rejected at time t for the selected Model A of Table 19.

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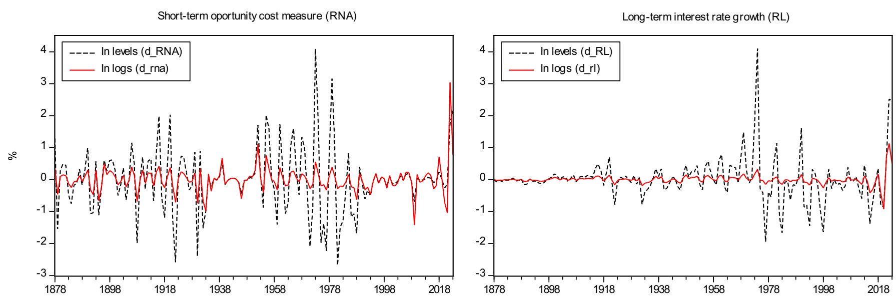

Figure B.3. Comparison of Interest Rates in Levels and Logarithms

Notes: Figure compares the growth rates of short-term opportunity cost (ΔRNA_t) and long-term interest rates (ΔRL_t) in logarithmic and level terms from 1878 to 2023. All values are reported as percentages.

Table B.1. Seminal UK Historical Competing Money Demand Estimates (1878 – 1970)

	Dependent variable: $\Delta(m-p)_t$							
Regressors	Hendry and Ericsson (1985)		Longbottom and Holly (1985)		Escribano (1985, 1986)		Hendry and Ericsson (1991)	
$\Delta^2(m-p)_{t-1}$	0.37	(0.05)	-	-	-	-	-	-
$\Delta^2(m-p)_{t-2}$	-0.06	(0.075)	-0.13	(0.041)	-	-	-0.10	(0.04)
$\Delta(m-p)_{t-1}$	-	-	0.47	(0.063)	0.45	(0.073)	0.45	(0.06)
$\Delta(m-p)_{t-2}$	-	-	-	-	-0.16	(0.053)	-	-
$\Delta(m-p)_{t-3}$	-	-	-	-	0.08	(0.047)	-	-
$(1/4)\Delta_4 y_t$	0.64	(0.14)	-	-	-	-	-	-
Δy_t	-	-	-	-	0.08	(1.7)	-	-
$\Delta^2 p_t$	-0.47	(0.040)	-0.63	(0.041)	-	-	-	-
$\Delta^2 p_{t-2}$	-0.14	(0.07)	-	-	-	-	-	-
Δp_t	-	-	-	-	-0.61	(0.044)	-0.60	(0.04)
Δp_{t-1}	-	-	-0.22	(0.046)	0.37	(0.052)	0.39	(0.05)
Δp_{t-2}	-	-	-	-	-	-	-	-
ΔRL_t	-	-	-	-	-0.01	(0.006)	-	-
$(1/2)\Delta_2 RL_t$	-3.3	(1.1)	-	-	-	-	-	-
$\Delta_2 rl_t$	-	-	-0.011	(0.002)	-	-	-0.062	(0.021)
Δrs_t	-	-	-	-	-	-	-0.021	(0.006)
ΔRS_t	-	-	-	-	-0.008	(0.002)	-	-
$(m-p-y)_{t-4}$	-0.20	(0.02)	-	-	-	-	-	-
$(m-p)_{t-1}$	-	-	-0.058	(0.015)	-	-	-	-
y_t	-	-	0.065	(0.015)	-	-	-	-
RS_t	-0.78	(0.18)	-	-	-	-	-	-
rs_t	-	-	-0.0056	(0.004)	-	-	-	-
$D1_t$	1.9	(0.79)	-	-	0.04	(0.009)	-	-
$D2_t$	3.6	(0.6)	-	-	-	-	-	-
$D3_t$	0.6	(0.86)	-	-	0.04	(0.007)	-	-
$(D1+D3)_t$	-	-	0.034	(0.006)	-	-	0.037	(0.006)
$D4_t * \Delta RS_t$	-	-	-	-	-	-	0.080	(0.027)
DC_t	-	-	-	-	-	-	0.050	(0.008)
$\hat{u}_{Rt-1}$	-	-	-	-	-0.018	(0.03))	-	-
${\hat{u}_{Rt-1}}^2$	-	-	-	-	0.5	(0.15)	-	-
${\hat{u}_{Rt-1}}^3$	-	-	-	-	-2.18	(0.95)	-	-
$(\hat{u}_{Rt-1}-0.2){\hat{u}_{Rt-1}}^2$	-	-	-	-	-	-	-2.55	(0.59)
Constant	-0.086	(0.12)	-0.074	(0.031)	0.004	(0.003)	0.005	(0.002)
R^2	0.82		0.86		0.87		0.87	
$100 * \hat{\sigma}$	1.70%		1.46%		1.46%		1.42%	
No. of parameters	12		10		14		9	

Notes: Each column of the short-run equations presents coefficients obtained from separate OLS regressions, with standard errors provided in parentheses. These results are referenced in the papers cited above. Below are the details of the models:

(i) Hendry and Ericsson (1985), and Longbottom and Holly (1985)`s models are one-step EC.

(ii) Escribano (1985, 1986), and Hendry and Ericsson (1991)`s models are two-step NEC with cointegrating residual $\hat{u}_{Rt-1} = (m-p-y)_{t-1} + 0.31 + 7.0\ RS_{t-1}$.