Inflation control in a CVAR model with an application the Burns/Miller period in the USA

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Abstract

The idea is to show in the cointegrated VAR model how to make a nonstationary target variable stationary by controlling a monetary instrument. We show that a necessary condition for this is non-neutrality between instrument and target variables expressed as a non-zero element in the long-run impact matrix. We also show that if the monetary instrument is cointegrated with an intermediate target variable, then the latter can also be used as an instrument. An application to US data covering the Burns/Miller periods finds a significant, but positive, long-run impact on inflation rate from a shock to the federal funds rate. We find that the federal funds rate and the 3 month tbill rate are cointegrated, so the tbill rate would qualify as an intermediate instrument. We also find that the long-run impact on inflation from a shock to the tbill rate is significant and that it has a negative impact on inflation provided the spread between the Tbill rate and the federal funds rate is positive. Altogether we find only weak support for the widely held belief that the Federal Reserve Bank can bring US CPI inflation down by raising the federal funds rate.

Keywords: Vector autoregression, cointegration, control rule, instrument, monetary policy, federal funds rate.

JEL Classification: C3, E5

1 Introduction

Most literature on policy rules either assume that the variables are stationary or allow for nonstationarity, but do not consider whether this may have consequences for controllability.

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See for example the collected papers in Taylor (1999), Clarida, Gali and Gertler (1998) and references therein, where the economy is usually estimated as a VAR model, but without examining the conditions on the VAR parameters would render the ultimate (nonstationary) goal variables controllable given the available instruments.

We wish to address these questions relying on the following basic assumptions: (1) the VAR model is capable of satisfactorily describing the dynamics of the data, (2) the central bank is free to change the value of its instrument, (3) the central bank changes the value of its instruments based on a linear control rule which takes into account the target variable and the state of the economy, and (4) the 'economy' variables, i.e. the non-instrument variables, react to the central bank intervention.

One difficulty when estimating the effect of a central bank policy intervention on the target inflation rate is that the time it takes can be long and varying. Another difficulty is that interventions take place on a daily basis, but inflation rate is only measured on a monthly basis. Therefore, the evaluation of the final effect of a monetary intervention on the goal variable is empirically difficult. Since an intermediate variable - such as a market determined interest rate - is likely to respond fast to a policy intervention, the conduct of monetary policy has often been assessed based on an analysis of how an intermediate target, rather than the monetary policy instrument, affects the goal variable. We show that this can be a valid procedure, provided that the intermediate target can be controlled by the central bank and that the intermediate target cointegrates with the final target.

The control theory presented here is a modified version of Johansen and Juselius (2001). Chevillon and Kurita (2023) apply the ideas to address how to control the world temperature using CO_2 as an instrument. Castle and Kurita (2024) use them to discuss the implications of monetary policy when the policy rule includes digital assets, such as cryptocurrencies. Finally, Boug, Hungnes, Kurita (2024) ask the question whether the central bank can use the policy rate to stabilize house prices and housing debt.

The paper is structured as follows: In the first part, we illustrate the concepts and the results using a simple policy control rule for a VAR(1) model. We derive the theoretical conditions on the parameters of the VAR model under which a goal variable is controllable. For a nonstationary inflation rate we find the necessary condition for controllability to be a significant long-run impact from shocks to the instrument variable on the target variable. Given controllability, we derive a suitable control rule for the instrument variable with the following property: when the control rule is applied at all points in time, a nonstationary target variable will become stationary with a desired mean. The theoretical results are illustrated with an application to the Burns/Miller regime in the USA.

2 Definition of the control problem in the CVAR(1) model

2.1 The CVAR(1) model and the long-run value

The model is defined by the equations

$$\Delta x_{t+1} = \alpha(\beta' x_t - \mu) + \varepsilon_{t+1}, \tag{1}$$

where ε_t are are p vectors and i.i.d. $(0, \Omega)$, α and β are $p \times r$ matrices, and μ is an r vector, see Johansen (1996). The equations define a cointegrated $I(1)$ process CVAR(1) with r cointegrating relations β, if and only if the eigenvalues of $\beta'\alpha$ satisfy the $I(1)$ condition

$$|1 + eig(\beta'\alpha)| < 1. \tag{2}$$

This condition, which will be assumed throughout, implies that $\beta'\alpha$, and therefore $\alpha'_\perp \beta_\perp$, is non-singular and we define the long-run matrix by

$$C = \beta_\perp(\alpha'_\perp \beta_\perp)^{-1}\alpha'_\perp = I_p - \alpha(\beta'\alpha)^{-1}\beta'. \tag{3}$$

By the Granger representation, x_t is represented as a nonstationary process

$$x_t = C\sum_{i=1}^{t} \varepsilon_i + y_t + A + \alpha(\beta'\alpha)^{-1}\mu, \tag{4}$$

where A depends on initial values ($\beta' A = 0$) and y_t is stationary with mean zero such that x_t is cointegrated, that is, $\beta' x_t = \beta' y_t + \mu$ is stationary with mean μ. Thus x_t is a CVAR(1) process and condition (2) rules out unit roots and explosive roots.

The following result defines the long-run value of the process, that is, the value x_{t+h} would converge towards, if x_t is kept fixed and the errors were switched off, see Figure 1:

Lemma 1 *For the $I(1)$ process x_t given by (1), the expectation $E(x_{t+h}|x_t)$ converges to the long-run value of x_{t+h} starting at x_t, defined by*

$$L_\infty(x_t) = \lim_{h\to\infty} E(x_{t+h}|x_t) = Cx_t + \alpha(\beta'\alpha)^{-1}\mu, \tag{5}$$

which satisfies $\beta' L_\infty(x_t) = \mu$, such that $L(x_t)$ is a point in the attractor set (6), defined by

$$\{x|\beta' x = \mu\} = \alpha(\beta'\alpha)^{-1}\mu + sp(\beta_\perp). \tag{6}$$

Note that $\alpha'_\perp(E(x_{t+h}|x_t) - L_\infty(x_t)) = 0$, so that if the errors were switched off, x_{t+h} would move towards the long-run value in the direction of α.

Proof of Lemma 1. *Proof of (5)*: From equation (1) we find for given x_t, that

$$x_{t+h} = (I_p + \alpha\beta')^h x_t + \sum_{i=0}^{h-1}(I_p + \alpha\beta')^i(\varepsilon_{t+i} - \alpha\mu),$$

such that

$$E(\alpha'_\perp x_{t+h}|x_t) = \alpha'_\perp x_t,$$

$$E(\beta' x_{t+h}|x_t) = (I_r + \beta'\alpha)^h \beta' x_t - \sum_{i=0}^{h-1} (I_r + \beta'\alpha)^i \beta'\alpha\mu \to -\sum_{i=0}^{\infty} (I_r + \beta'\alpha)^i \beta'\alpha\mu = \mu,$$

which combine to

$$E(x_{t+h}|x_t) = \beta_\perp(\alpha'_\perp\beta_\perp)^{-1} E(\alpha'_\perp x_{t+h}|x_t) + \alpha(\beta'\alpha)^{-1} E(\beta' x_{t+h}|x_t)$$
$$\to Cx_t + \alpha(\beta'\alpha)^{-1}\mu = L_\infty(x_t), \text{ for } h \to \infty.$$

■

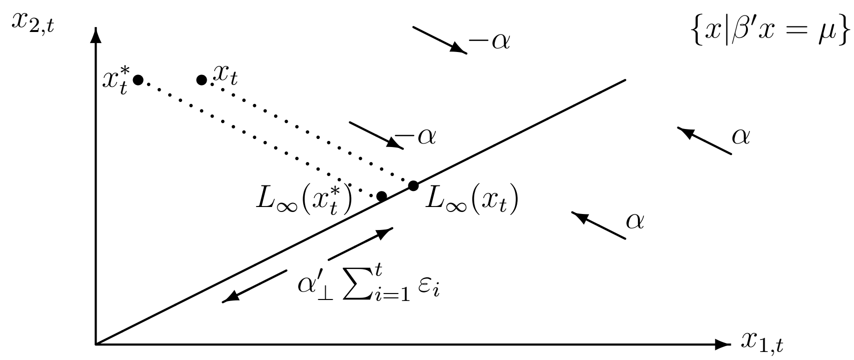

Figure 1: The plot shows the bivariate process $x_t = (x_{1,t}, x_{2,t})$, and the long-run value $L_\infty(x_t) = Cx_t + \alpha(\beta'\alpha)^{-1}\mu$. If the errors were switched off, the process would move from x_t along the direction of $-\alpha$ to the point $L_\infty(x_t)$. The adjustment vectors $\pm\alpha$ pull the process towards the attractor set, with a force that depends on the magnitude of the distance $\beta' x_t - \mu$. Thus, for points on the attractor set the force is zero, and there is no tendency to move away. Such points are called equilibrium or long-run values. The common trends $\alpha'_\perp \sum_{i=1}^t \varepsilon_i$ push the process parallel to the attractor set and generate the nonstationary behavior of the process. If x_t is shifted to x_t^*, the long-run value is shifted along the attractor set. It is this effect that allows to control $x_{2,t}$ by changing the value of $x_{1,t}$.

2.2 The control problem

This section introduces the control problem in the CVAR(1) and leaves the treatment of the general model with several lags and a trend to the results in Appendix.

As a stylized example we discuss a policy control situation, where the central bank sets the value of the central bank instrument (the federal funds rate) as a reaction to the current state of the economy, in order to make a nonstationary target variable (the inflation rate) stationary around a given value. Our aim is to formalize the impact on the dynamics of the system, when the central bank applies such a policy control rule at all points of time.

Definition 1 *We let a, b, c denote $p-$dimensional unit vectors and let κ be a $p-$vector and κ^* a real number.*

(i) An instrument variable $a'x_t$, has the property that its value can be changed by an intervention, so that x_t can replaced by

$$x_t + a(\kappa' x_t - \kappa^*), \tag{7}$$

for any κ and κ^*.

(ii) The target variable, $b'x_t$, is the variable one would like to control using an instrument $a'x_t$, so that $b'x_t$ becomes stationary with mean b^*, the desired target value.

(iii) An intermediate target variable $c'x_t$, is a variable that is cointegrated with the target $b'x_t$, so that $\psi b'x_t + c'x_t$ is stationary for some non-zero number ψ.

(iv) By repeated application of the control rule (7) for all t, x_t is changed to the process x_{t+1}^{new} given by the equation

$$x_{t+1}^{new} = (I_p + \alpha\beta')(x_t^{new} + a(\kappa' x_t^{new} - \kappa^*)) - \alpha\mu + \varepsilon_{t+1}. \tag{8}$$

Below we prove the following results. In Theorem 1 we find the properties of the new process x_t^{new} for a given control rule (κ, κ^*), without specifying a target variable, and find the condition under which x_t^{new} is a CVAR(1).

In Corollary 1 we present, for a given target $b'x_t$ and target value b^*, a control rule which controls $b'x_t$ using the instrument $a'x_t$, such that if $b'Ca \neq 0$, $b'x_t$ becomes stationary with mean b^*, and in Corollary 2 we use an intermediate target, $c'x_t$, to control a target $b'x_t$. In Theorem 2 we discuss controlling a stationary target variable, using a control which only depends on κ^*.

Theorem 1 *Let x_t be given by $\Delta x_{t+1} = \alpha(\beta' x_t - \mu) + \varepsilon_{t+1}$, where the parameters satisfy the $I(1)$ condition $|1 + eig(\beta'\alpha)| < 1$. The new process, see equation (8), is given by the VAR model:*

$$\Delta x_{t+1}^{new} = \alpha_\dagger(\beta_\dagger' x_t^{new} - \mu_\dagger) + \varepsilon_{t+1}, \tag{9}$$

where

$$\alpha_\dagger = (\alpha, a)\begin{pmatrix} I_r & \beta' a \\ 0 & 1 \end{pmatrix}, \ \beta_\dagger = (\beta, \kappa), \text{ and } \mu_\dagger = \begin{pmatrix} \mu \\ \kappa^* \end{pmatrix}. \tag{10}$$

The $I(1)$ condition for x_t^{new} becomes

$$|1 + eig\begin{pmatrix} \beta'\alpha & (I_r + \beta'\alpha)\beta' a \\ \kappa'\alpha & \kappa'(I_p + \alpha\beta')a \end{pmatrix}| < 1, \tag{11}$$

which is satisfied if

$$\kappa'\alpha = 0, \ \ and \ |1+\kappa' a| < 1. \tag{12}$$

If the $I(1)$ *condition is satisfied, then* x_t^{new} *is a CVAR(1) with* $(\beta,\kappa)'x_t^{new}$ *stationary with mean* $\mu_\dagger$, *and*

$$\kappa' C a \neq 0. \tag{13}$$

The long-run value of x_{t+h}^{new} *is given by*

$$L_\infty(x_t^{new}) = C_\dagger x_t^{new} + \alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger,$$

where,

$$C_\dagger = C - Ca(\kappa' Ca)^{-1}\kappa' C, \tag{14}$$

$$\alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger = (I_p - Ca(\kappa' Ca)^{-1}\kappa')\alpha(\beta'\alpha)^{-1}\mu + Ca(\kappa' Ca)^{-1}\kappa^*. \tag{15}$$

Proof of Theorem 1. *Proof of (9), (10), (11) and (12):* From (8) we find the equation for x_{t+1}^{new} in error correction form

$$\Delta x_{t+1}^{new} = ((I_p+\alpha\beta')(I_p + a\kappa') - I_p)x_t^{new} - (I_p+\alpha\beta')a\kappa^* - \alpha\mu + \varepsilon_{t+1} \tag{16}$$

$$= (\alpha, a)\begin{pmatrix} I_r & \beta' a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \beta' x_t^{new} - \mu \\ \kappa' x_t^{new} - \kappa^* \end{pmatrix} + \varepsilon_{t+1} = \alpha_\dagger(\beta_\dagger' x_t^{new} - \mu_\dagger) + \varepsilon_{t+1},$$

such that (9), (10), and (11) hold. If $\kappa'\alpha = 0$, and $|1+\kappa' a| < 1$, then

$$1 + eig\begin{pmatrix} \beta'\alpha & (I_r+\beta'\alpha)\beta' a \\ \kappa'\alpha & \kappa'(I_p+\alpha\beta')a \end{pmatrix} = eig\begin{pmatrix} 1+\beta'\alpha & (I_r+\beta'\alpha)\beta' a \\ 0 & 1+\kappa' a \end{pmatrix}.$$

The absolute value of the eigenvalues are $|eig(1+\beta'\alpha)|$ and $|1+\kappa' a|$ which are assume less than one.

Proof of (13): If $|1 + eig\left(\beta_\dagger'\alpha_\dagger\right)| < 1$, the eigenvalues of $\beta_\dagger'\alpha_\dagger$ have negative real part, and are therefore non-zero, such that $\det(\beta_\dagger'\alpha_\dagger) \neq 0$, and similarly by (2), $\det(\beta'\alpha) \neq 0$. We then find

$$\det(\beta_\dagger'\alpha_\dagger) = \det\begin{pmatrix} \beta'\alpha & (I_r+\beta'\alpha)\beta' a \\ \kappa'\alpha & \kappa'(I_p+\alpha\beta')a \end{pmatrix}$$

$$= \det(\beta'\alpha)(\kappa'(I_p+\alpha\beta')a - \kappa'\alpha(\beta'\alpha)^{-1}(I_r+\beta'\alpha)\beta' a)$$

$$= \det(\beta'\alpha)(\kappa'(I_p+\alpha\beta' - \alpha(\beta'\alpha)^{-1}\beta' - \alpha\beta')a) = \det(\beta'\alpha)\kappa' Ca,$$

such that $\kappa' Ca \neq 0$.

Proof of (14) and (15): The long-run matrix for x_t^{new} is

$$C_\dagger = I_p - (\alpha, a)((\beta,\kappa)'(\alpha, a))^{-1}(\beta,\kappa)', \tag{17}$$

and we find from (27) and (28) in the Appendix, Lemma 2, that (14) and (15) hold. ■

It follows from Theorem 1, that if we want to control a nonstationary target variable $b'x_t$ with target value b^*, we can simply use the control rule $(\kappa, \kappa^*) = (b, b^*)$, but only if the $I(1)$ condition (11) for x_t^{new} holds for this choice. But we give in Corollary 1 a different choice of (κ, κ^*), based on the construction in Figure 2, which always satisfies the $I(1)$ condition.

The idea in Figure 2 is to move x_t by $a(\kappa' x_t - \kappa^*)$ such that the long-run value of $b'x_{t+h}$ (5), starting at $x_t + a(\kappa' x_t - \kappa^*)$ is b^*, if we do not use any more controls, that is, we suggest to find (κ, κ^*) such that

$$b'L(x_t) = b'C(x_t + a(\kappa' x_t - \kappa^*)) + b'\alpha(\beta'\alpha)^{-1}\mu = b^*.$$

This determines κ by $b'C + b'Ca\kappa' = 0$, or $\kappa' = -(b'Ca)^{-1}b'C$, which satisfies (12) because $\kappa'\alpha = 0$ and $|1 + \kappa' a| = 0$. This result is given in Corollary 1.

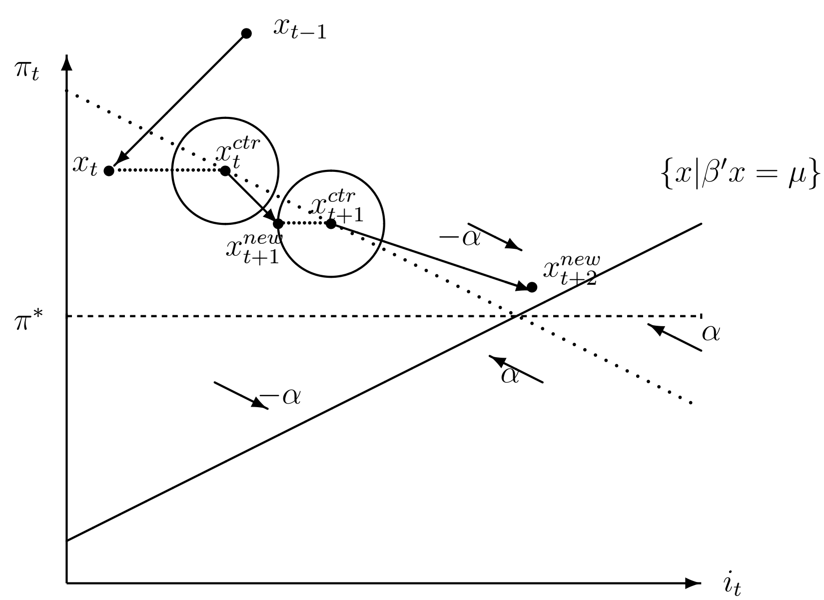

Figure 2: An illustration, using interest rate, i_t, and inflation rate, π_t, of the idea of repeatedly applying a control rule. At time t, the CVAR(1) produces x_t by taking into account the information x_{t-1}. At this point the bank intervenes and replaces i_t by i_t^{ctr}, which is here chosen such that the long-run value, of x_t starting from x_t^{ctr} would be π^*. At time $t+1$ the CVAR produces the new value of the process x_{t+1}^{new}, by taking into account the information $x_t^{ctr} = (i_t^{ctr}, \pi_t)'$. The circle indicates that an error term ε_{t+1} is added to construct the new value. By repeatedly shifting the process to the set of points where the long-run value of π_t is π^*, we change π_t to a stationary process π_t^{new} with mean π^*, see Theorem 1.

Corollary 1 *We consider a nonstationary target $b'x_t$ and a target value b^*. We assume $b'Ca \neq 0$, that is, there is no long-run neutrality of the instrument $a'x_t$ to the target $b'x_t$.*

We choose (κ,κ^*) *by solving the equation, as in Figure 1,*

$$b'L_\infty(x_t + a(\kappa' x_t - \kappa^*)) = b^*, \tag{18}$$

which gives

$$\kappa' = -(b'Ca)^{-1}b'C \quad \text{and } \kappa^* = (b'Ca)^{-1}(b'\alpha(\beta'\alpha)^{-1}\mu - b^*). \tag{19}$$

Applying this control rule at all times generates x_t^{new}*, which by Theorem 1 is a CVAR(1) with cointegrating space* $sp(\beta, b)$ *and adjustment space* $sp(\alpha, a)$*, and the intervention can be expressed as*

$$\kappa' x_t^{new} - \kappa^* = (b'Ca)^{-1}(b'\alpha(\beta'\alpha)^{-1}(\beta' x_t^{new} - \mu) - (b' x_t^{new} - b^*)), \tag{20}$$

which is a linear combination of the two stationary processes $\beta' x_t^{new} - \mu$ *and* $b' x_t^{new} - b^*$*, which have mean zero.*

Proof of Corollary 1. Using Lemma 1, equation (18) becomes

$$b'C(x_t + a(\kappa' x_t - \kappa^*)) + b'\alpha(\beta'\alpha)^{-1}\mu = b^*.$$

This implies that $b'C + b'Ca\kappa' = 0$ and $-b'Ca\kappa^* + b'\alpha(\beta'\alpha)^{-1}\mu = b^*$, giving the solution (19) when $b'Ca \neq 0$. We note that $\kappa'\alpha = -(b'Ca)^{-1}b'C\alpha = 0$ and $\kappa' a = -(b'Ca)^{-1}b'Ca = -1$, such that (12) is satisfied, and Theorem 1 holds.

To prove (20) we use $Cx_t^{new} = x_t^{new} - \alpha(\beta'\alpha)^{-1}\beta' x_t^{new}$, such that

$$\begin{aligned}\kappa' x_t^{new} - \kappa^* &= -(b'Ca)^{-1}(b'Cx_t^{new} - b'\alpha(\beta'\alpha)^{-1}(\mu - b^*),\\ &= (b'Ca)^{-1}(b'\alpha(\beta'\alpha)^{-1}(\beta' x_t^{new} - \mu) - (b' x_t^{new} - b^*)).\end{aligned}$$

■

Theorem 2 *If the target variable* $b'x_t$ *is stationary and the target value is* b^**, we choose* $(\kappa,\kappa^*) = (0,\kappa^*)$ *such that* x_t^{new} *is given by*

$$\Delta x_{t+1}^{new} = \alpha\beta' x_t^{new} - (I_p + \alpha\beta')a\kappa^* - \alpha\mu + \varepsilon_{t+1}, \tag{21}$$

with cointegration space $sp(\beta)$*. It follows that* $\alpha_\perp' x_{t+1}^{new}$ *is a random walk with a trend* $-\alpha_\perp' a\kappa^* t$*, when* $\kappa^* \neq 0$*, and* $b' x_t^{new}$ *is stationary with mean*

$$E(b' x_t^{new}) = b^*,$$

if $b'\alpha(\beta'\alpha)^{-2}\beta' a \neq 0$*, and*

$$\kappa^* = (b^* - E(b'x_t))/b'\alpha(\beta'\alpha)^{-2}\beta' a.$$

Proof of Theorem 2. For $\kappa = 0$, we find from (8) equation (21) for x_t^{new}. Thus the new process x_t^{new} is a CVAR(1) process with cointegrating space $sp(\beta)$. Multiplying by $\alpha_\perp'$ we find that $\alpha_\perp' x_t^{new}$ is a random walk with trend $-\alpha_\perp a\kappa^* t$, when $\kappa^* \neq 0$. Multiplying by β' we find

$$E(\beta' x_t^{new}) = \mu + (\beta'\alpha)^{-1}\beta'(I_p + \alpha\beta')a\kappa^* = E(\beta' x_t) + (\beta'\alpha)^{-1}\beta'(I_p + \alpha\beta')a\kappa^*.$$

Because $b'x_t$ is stationary, $b = \beta\phi$, for an r vector $\phi' = b'\alpha(\beta'\alpha)^{-1}$, such that multiplying by ϕ' we find

$$\begin{aligned} E(b'x_t^{new}) - E(b'x_t) &= b'\alpha(\beta'\alpha)^{-2}\beta'(I_p + \alpha\beta')a\kappa^* = b'\alpha(\beta'\alpha)^{-2}\beta' a\kappa^* + b'\alpha(\beta'\alpha)^{-1}\beta' a\kappa^* \\ &= b'\alpha(\beta'\alpha)^{-2}\beta' a\kappa^* + b'(I_p - C)a\kappa^* = b'\alpha(\beta'\alpha)^{-2}\beta' a\kappa^*, \end{aligned}$$

because $b'a = 0$, $b'C = 0$. It follows that

$$E(b'x_t^{new}) - b^* = E(b'x_t) - b^* + b'\alpha(\beta'\alpha)^{-2}\beta' a\kappa^*,$$

which equals zero, if we can choose $\kappa^* = (b^* - E(b'x_t))/b'\alpha(\beta'\alpha)^{-2}\beta' a$. ■

Corollary 2 *Let $a'x_t$ be an instrument, let $b'x_t$ be the nonstationary final target with target value b^*, and $b'Ca \neq 0$, and let $c'x_t$ be an intermediate target, that is, we assume that for some $\psi \neq 0$, $\psi b'x_t + c'x_t$ is stationary with mean τ, see Definition 1 (iii). Then the control rule*

$$\kappa' = -(c'Ca)^{-1}c'C \quad \text{and } \kappa^* = (c'Ca)^{-1}(c'\alpha(\beta'\alpha)^{-1}\mu - \tau + \psi b^*), \tag{22}$$

makes $b'x_t^{new}$ stationary with mean τ.

Proof of Corollary 2. We use the decomposition $I_p = C + \alpha(\beta'\alpha)^{-1}\beta'$, and find

$$\begin{aligned} \psi b'x_t^{new} &= (\psi b'x_t^{new} + c'x_t^{new}) - c'x_t^{new} \\ &= (\psi b'x_t^{new} + c'x_t^{new}) - c'Cx_t^{new} - c'\alpha(\beta'\alpha)^{-1}\beta' x_t^{new} \\ &= (\psi b'x_t^{new} + c'x_t^{new}) + (c'Ca)(\kappa' x_t^{new} - \kappa^*) + (c'Ca)\kappa^* - c'\alpha(\beta'\alpha)^{-1}\beta' x_t^{new} \end{aligned} \tag{23}$$

Taking expectations of the individual terms, we find using that $\psi b'x_t^{new} + c'x_t^{new}$ and $\psi b'x_t + c'x_t$ are stationary with the same mean:

$$E(\psi b'x_t^{new} + c'x_t^{new}) = E(\psi b'x_t + c'x_t) = \tau.$$

Moreover, see (19) in Corollary 1, $\kappa' x_t^{new} - \kappa^*$ is stationary with expectation zero, and finally $E(c'\alpha(\beta'\alpha)^{-1}\beta' x_t^{new}) = c'\alpha(\beta'\alpha)^{-1}\mu$, such that, using the expression (22) for κ^*,

$$\psi E(b'x_t^{new}) = \tau + 0 + (c'\alpha(\beta'\alpha)^{-1}\mu - \tau + \psi b^*) - c'\alpha(\beta'\alpha)^{-1}\mu = \psi b^*,$$

which implies that $E(b'x_t^{new}) = b^*$. ■

3 Summarizing the main results

We have investigated the CVAR(1) $\Delta x_{t+1} = \alpha(\beta' x_t - \mu) + \varepsilon_t$, with the purpose of formulating a control problem as that of making a nonstationary target variable, $b'x_t$, stationary around

Table 1: The monetary transmission mechanism

Central bank	*The market*				
↓			↙ ↘		
$\overset{(+)}{R_{CB}}$ ⟶	$\overset{(+)}{R_s}$ ⟶	$\overset{(+)}{R_l}$ ⟶	$\overset{(-)}{y-y^*}$ ⟶	$\overset{(-)}{un}$ ⟶	$\overset{(-)}{\Delta p}$

a target value b^* using an instrument variable $a'x_t$. We have defined a general control rule $x_t^{ctr} = x_t + a(\kappa' x_t - \kappa^*)$ which, when applied repeatedly, leads to a modified CVAR(1) $\Delta x_{t+1}^{new} = \alpha_\dagger(\beta_\dagger' x_t^{new} - \mu_\dagger) + \varepsilon_{t+1}$, for which $\kappa' x_t^{new} - \kappa^*$ is stationary with mean zero.

These results show that if $b'Ca \neq 0$, and we choose $\kappa' = -(b'Ca)^{-1}b'C$ and $\kappa^* = (b'Ca)^{-1}(b'\alpha(\beta'\alpha)^{-1}\mu - b^*)$, then $b'x_t^{new} - b^*$ is stationary with mean zero. If instead $b'x_t$ is stationary, we choose $\kappa' = 0$, and for $b'\alpha(\beta'\alpha)^{-2}\beta' a \neq 0$, and $\kappa^* = (b^* - E(b'x_t))/b'\alpha(\beta'\alpha)^{-2}\beta' a$, the mean of $b'x_t^{new}$ becomes b^*.

The results can be generalized to the model with more lags an a linear trend, as is done in the Appendix.

4 A simple monetary transmission mechanism

A standard version of the monetary transmission mechanism describes how changes in the Central Bank policy rate (R_{CB}) - dynamically affect the domestic economy through the subsequent adjustment of the short-term interest rate, R_s, long-term interest rates (R_l), the output gap ($y - y^*$), the unemployment rate (un) and finally the goal variable, the inflation rate (Δp). The following simple diagram serves as an illustration:

To keep the illustration as simple as possible, we focus here on how a shock to the policy rate transmits to the short-term interest rate and then to the goal variable, the inflation rate. The price of this simplicity is of course to some extent a lack of realism. However, we have performed a CVAR analysis of the fully specified model and all basic conclusions of the small model still apply. While might seem surprising, it can be understood by noting that cointegration properties are robust to changes in the information set.

5 An empirical illustration

The empirical task is to study monetary policy in the USA over the period 1970:1 to 1979:5 when the Federal Reserve Bank was presided by Arthur F. Burns 1970-1978 and William Miller. This period has obtained a renewed interest among economists and policy makers because inflation rates in the aftermath of the Corona pandemic started to increase to levels that resembled the ones in the seventies. The historical narrative is that the central bank then did not live up to its most important task of keeping the inflation low. When inflation rates rose after almost four decades of unprecedented low inflation rates, many asked whether

Table 2: VAR model specification

	$Norm$	$ARCH$	$p-r$	λ_i	$Trace^*$	c_{95}	$p-val$	$w.e$	$u.v$
ΔFf_t	1.46[0.48]	0.41[0.81]	3	0.29	57.90	35.07	0.00	6.28 [0.04]	19.70 [0.00]
$\Delta Tb3m_t$	0.49[0.78]	5.30[0.07]	2	0.18	22.13	20.16	0.03	24.36 [0.00]	3.95 [0.05]
$\Delta^2 p_t$	1.04[0.60]	0.78[0.68]	1	0.01	1.43	9.14	0.87	33.23 [0.00]	0.86 [0.35]

p-values in square brackets

the Central Bank had again failed as in the seventies. But was inflation at all controllable in the seventies? We shall use the theoretical framework in Section 2 to address this question.

5.1 Defining the model

The CVAR model with two lags is defined as follows:

$$\Delta x_t = \alpha(\beta' x_{t-1} - \mu) + \Gamma_1 \Delta x_{t-1} + \Phi D_t + \varepsilon_t, \tag{24}$$

where ε_t is $p \times 1$ distributed as i.i.d. $(0, \Omega)$, α and β are $p \times r$ matrices, μ is an $r \times 1$ vector, and $x_t' = [Ff, Tb3, \Delta p]_t$ where Ff_t is the federal funds rate, $Tb3_t$ is the $Tb3$ rate, and Δp the first difference of log of the monthly CPI. The annual interest rates in % are monthly averages of daily observations, which have been divided by 1200 to achieve comparability with the inflation rate. The seventies witnessed two major oil crises in 1973 and 1979. The first one is modelled by a dummy variable defined as $Dtr73.08_t = 1$ for $t =$ 1973:8 and = -0.5 for $t =$ 1973:7 and 1973:9, the second by leaving out the last half of 1979 so that not to end the sample in a period of great turmoil. Additionally, we need to include two intervention dummies $Dp71.08 = 1$ for $t = 1971:8$ and $Dp74.08 = 1$ for $t = 1974:08$ to control for a large increase in $Tb3$ and in the Ff rate. To sum up $D_t = [Dp71.08, Dtr73.08, Dp74.08]_t$. All data series are from St. Louis Bank database. Graphs of the data are given in Appendix A.

5.2 Specification, rank and general properties

The theoretical results of Section 2 were derived under the assumption that the CVAR model is a satisfactory description of the data. Table 2 reports tests of residual normality and homoscedasticity (ARCH). All empirical calculations are based on the software package CATS in RATS (Hansen and Juselius, 1994).

The first part of Table 2 reports the Jarque-Bera normality and the ARCH(2) tests for each equation and shows that the VAR model is a satisfactory description of the data. The second part reports the trace tests for the determination of the cointegration rank. The eigenvalues, λ_i, the Bartlett corrected trace test and 95% percentiles of the asymptotic distribution shows that $p-r = 1$ is strongly supported by the data. Thus, the system can be

Table 3: The pulling forces

	Ff_t	$Tb3_t$	Δp_t	$const$
β_1'	1.00	-1.10 $-[21.3]$	0	0.00 [4.15]
β_2'	-1	0	1.00	0
	ΔFf_t	$\Delta Tb3_t$	$\Delta^2 p_t$	
α_1'	0.12 [2.5]	0.41 [5.4]	0.64 [2.2]	
α_2'	-0.02 $[-1.3]$	-0.03 $[-1.3]$	-0.65 $[-6.3]$	

t-values in square brackets

described by one pushing exogenous force and two pulling endogenous forces. To get a first view of these forces tests of weak exogeneity and unit vector for α for $r = 2$ are reported at r.h.s. of Table 2 for each variable. The weak exogeneity results show that the Ff rate is the variable that is closest to being weakly exogenous in this system, but a p-value of only 0.04 is a sign that the exogenous force may also contain shocks from the other variables. The unit vector in α results show that inflation rate is the only variable that seems to be purely adjusting. The 3 months treasury bill rate with a p-value of 0.05 indicates that it is partly adjusting to the system, partly pushing the system.

5.3 Estimating the pulling and pushing forces

The first part of Table 5.3 reports the two cointegration relations subject to two overidentified restrictions, accepted with a p-value of 0.82, and the corresponding adjustment coefficients.

The first cointegration relation, $\beta_1' x_t$, shows that the Ff rate and the $Tb3$ rate are cointegrated $(1, -1.1)$ and, hence, that the latter can act as an intermediate target variable. The question whether the spread, $Ff - Tb3$, can be accepted as stationary was borderline rejected based on the test statistic, $\chi^2(1) = 3.70[0.05]$. The α_1 coefficients show that equation $\Delta Tb3$ is significantly error-correcting, that equation ΔFf is error-increasing - but not very significantly so - and that the effect of the deviation $Ff - 1.1Tb3$ has a positive effect on $\Delta^2 p_t$. Thus, a positive shock to the Ff rate will lower inflation, but only if the $Tb3$ rate increases sufficiently to satisfy $Ff - 1.1Tb3 < 0$.

The second cointegration relation shows that the Ff rate and inflation rate are cointegrated $(1, -1)$ implying that the Fisher parity holds at least in the short end of the term structure. The α_2 coefficients show that only inflation is adjusting (consistent with the unit vector in α result of Table 2).

Table 4: The C matrix

	$\sum_{i=1}^{t}\varepsilon_{Ff_i}$	$\sum_{i=1}^{t}\varepsilon_{Tb3Ff_i}$	$\sum_{i=1}^{t}\varepsilon_{\Delta p_i}$
Ff_t	2.52 [4.79]	-0.88 [-1.97]	-0.07 [-0.90]
$Tb3Ff_t$	0.27 [4.79]	-0.09 [-1.97]	-0.01 [-0.90]
Δp_t	**2.72** [4.79]	**-0.95** [-1.97]	-0.08 [-0.90]

t-values in squared brackets

5.4 The pushing force

The MA representation of the CVAR model provides information about the pushing force of the system.

$$x_t = C\sum_{i=1}^{t}\varepsilon_i + C^*(L)\varepsilon_t + X_0$$

where $C = \beta_\perp(\alpha'_\perp(I-\Gamma_1)\beta_\perp)^{-1}\alpha'_\perp, C^*(L)\varepsilon_t$ stands for stationary short-run effects, and X_0 is a summary of constant terms.

The results above showed that there is both an interest rate spread and a level effect on inflation. Because both effects are relevant and interesting, the moving average representation is calculated based on the transformed vector, $x'_t = [Ff, Tb3Ff, \Delta p]_t$ where $Tb3Ff = Tb3 - Ff$.[1] The common trend, ct_t

$$ct_t = \alpha'_\perp\sum_{i=1}^{t}\varepsilon_i = \sum_{i=1}^{t}\varepsilon_{Ff_i} - \underset{[-1.87]}{0.35}\sum_{i=1}^{t}\varepsilon_{Tb3Ff_i} \underset{[-0.97]}{-0.03}\sum_{i=1}^{t}\varepsilon_{\Delta p_i}$$

shows that cumulated shocks to the Ff rate is the dominant force, but also that shocks to the spread between the $Tb3$ rate and the Ff rate has a negative (though only borderline significant) effect on the stochastic trend. Table 4 reports the estimates of the C matrix.

Whether inflation is controllable or not by the central bank, depends primarily on the significance of the long-run impact of a shock to the Ff rate, but also to the $Tb3$ rate (as it was cointegrated with the Ff rate). The estimates in the inflation row, show that there is a significant positive long-run effect on inflation from a shock to the Ff rate, ($C_{Ff,\Delta p} = 2.72$), and a negative effect from the spread ($C_{Tb3Ff,\Delta p} = 0.95$). Thus, a rise in the Ff rate is likely to have had a cost effect on inflation whereas a positive spread, $Tb3 - FF > 0$, had a dampening effect.

Because the long-term impact of a shock to the Ff rate on inflation is significant, repeatedly following the monetary policy rule (20) would make inflation stationary around a

[1] Note that this transformation is without loss of information.

pre-specified constant value, for example 2%. However, the positive sign of the federal funds coefficient makes such a rule difficult to sell to a central banker. It might suggest that the model is too simple, for example that the $Tb3$ rate alone cannot represent the economy-wide variables in the monetary transmission mechanism. However, adding the long-term bond rate, the unemployment rate, the excess M2 liquidity and the GDP growth rate to the variable set did not change the main results.

Perhaps, the conventional way of thinking about the monetary transmission mechanism may not be robust to how the nonstationarity of the data affects the monetary transmission mechanism.

6 Summary and concluding discussion

We have investigated the CVAR(1) $\Delta x_{t+1} = \alpha(\beta' x_t - \mu) + \varepsilon_t$, with the purpose of formulating a control problem as that of making a nonstationary target variable, $b'x_t$, stationary around a target value b^* using an instrument variable $a'x_t$. We have defined a general control rule $x_t^{ctr} = x_t + a(\kappa' x_t - \kappa^*)$ which, when applied repeatedly, leads to a modified CVAR(1) $\Delta x_{t+1}^{new} = \alpha_\dagger(\beta_\dagger' x_t^{new} - \mu_\dagger) + \varepsilon_{t+1}$, for which $\kappa' x_t^{new} - \kappa^*$ is stationary with mean zero.

These results show that if $b'Ca \neq 0$, and we choose $\kappa' = -(b'Ca)^{-1} b'C$ and $\kappa^* = (b'Ca)^{-1}(b'\alpha(\beta'\alpha)^{-1}\mu - b^*)$, then $b'x_t^{new} - b^*$ is stationary with mean zero. If instead $b'x_t$ is stationary, we choose $\kappa' = 0$, and for $b'\alpha(\beta'\alpha)^{-2}\beta' a \neq 0$, and $\kappa^* = (b^* - E(b'x_t))/b'\alpha(\beta'\alpha)^{-2}\beta' a$, the mean of $b'x_t^{new}$ becomes b^*. The results are generalized to the model with more lags and a linear trend.

We applied the theoretical results to the Burns-Miller period of the Federal Reserve Bank of the USA and demonstrated that the CVAR model with two lags was able to satisfactorily describe a system based on the federal funds rate, the 3 months treasury bill rate and the inflation rate. The empirical results demonstrated (1) that the federal funds rate was the primary exogenous variable with some minor effects from the treasury bill rate, (2) that inflation rate was exclusively adjusting in this system, that the spread, $Ff - Tb3$, was only borderline stationary with a p-value of 0.05, but that, $(Ff - 1.1Tb3)$ was clearly stationary and (3), that the long-run impact of a shock to the federal funds rate was highly significant but positive, suggesting that an increase in the policy rate tends to lift the term structure of interest rates and, thus, to have a cost-push effect on prices. The results also demonstrated a negative long-run impact on inflation from a shock to the spread $Tb3 - Ff$, implying that a positive spread had a dampening effect on inflation.

To conclude, the general belief that the Fed was able to lower inflation rate by increasing the Ff rate does not obtain empirical support based on the present information set.

7 APPENDIX A: The data

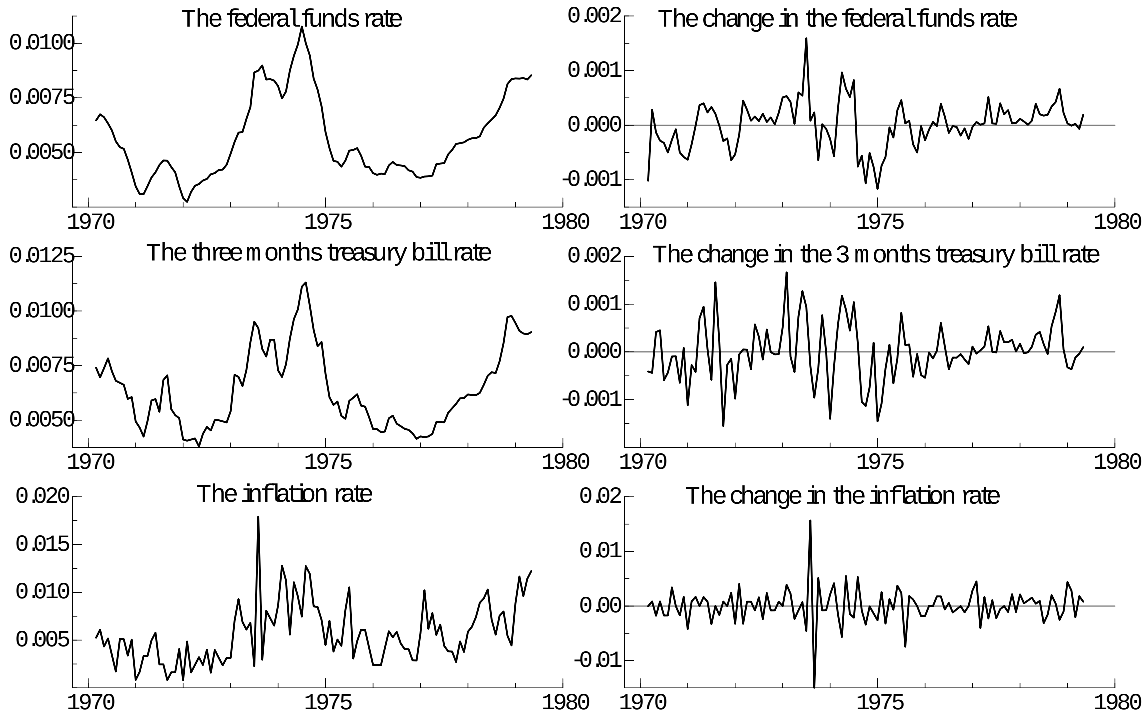

Figure 3: The graphs of the data in levels and differences. To have comparability with monthly inflation in log differences of CPI, interest rates are divided with 1200.

8 APPENDIX B:

In the Appendix B we first prove a Lemma on the inversion of a block matrix, which is used in the result in Theorem 1 on the long-run value of x_t^{new}. Next we formulate the control problem within the general CVAR(k) model and discusses a control rule, which depends on lagged variables, but only changes x_t and not the lagged values. This is done by expressing the CVAR(k) model in companion form, thereby reducing the control problem to a CVAR(1) and using the results in Section 2.1. In Corollary 3 we illustrate that we can use the simple idea in Figure 1 to construct a control rule depending on x_t and x_{t-1} for the model with two lags.

8.1 A technical Lemma

Lemma 2 *Let x_t be a CVAR(1) process and consider the control rule $\kappa' x_t - \kappa^*$, see Theorem 1, where we assume $rank(\beta'\alpha) = r$, and $\kappa' C a \neq 0$, then*

$$\begin{pmatrix} \beta'\alpha & \beta' a \\ \kappa'\alpha & \kappa' a \end{pmatrix}^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{25}$$

where

$$
\begin{aligned}
A_{11} &= (\beta'\alpha)^{-1} + (\beta'\alpha)^{-1}\beta' a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1},\\
A_{21} &= -(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1},\\
A_{12} &= -(\beta'\alpha)^{-1}\beta' a(\kappa' Ca)^{-1},\\
A_{22} &= (\kappa' Ca)^{-1}.
\end{aligned} \tag{26}
$$

It follows that the long-run value of x_t^{new} *is* $C_\dagger x_t^{new} + \alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger$ *for*

$$
C_\dagger = I_p - (\alpha, a)\begin{pmatrix} \beta'\alpha & \beta' a \\ \kappa'\alpha & \kappa' a \end{pmatrix}^{-1}(\beta,\kappa)' = C - Ca(\kappa' Ca)^{-1}\kappa' C, \tag{27}
$$

$$
\alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger = (I_p - Ca(\kappa' Ca)^{-1}\kappa)\alpha(\beta'\alpha)^{-1}\mu + Ca(\kappa' Ca)^{-1}\kappa^*. \tag{28}
$$

Note that the long-run value of $\kappa' x_t^{new}$ *is*

$$
\kappa'(C_\dagger x_t^{new} + \alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger) = \kappa' Ca(\kappa' Ca)^{-1}\kappa^* = \kappa^*.
$$

Proof of Lemma 2. *Proof of (26)*: Multiplying the matrices in (25), we find using $C = I_p - \alpha(\beta'\alpha)^{-1}\beta'$ that

$$
\begin{aligned}
\beta'\alpha A_{11} + \beta' a A_{21} &= I_r + \beta' a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1} - \beta' a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1} = I_r,\\
\beta'\alpha A_{12} + \beta' a A_{22} &= -\beta' a(\kappa' Ca)^{-1} + \beta' a(\kappa' Ca)^{-1} = 0,\\
\kappa'\alpha A_{12} + \kappa' a A_{22} &= -\kappa'(I_p - C)a(\kappa' Ca)^{-1} + \kappa' a(\kappa' Ca)^{-1} = \kappa' Ca(\kappa' Ca)^{-1} = 1,\\
\kappa'\alpha A_{11} + \kappa' a A_{21} &= \kappa'\alpha(\beta'\alpha)^{-1} + \kappa'\alpha(\beta'\alpha)^{-1}\beta' a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1} - \kappa' a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1}\\
&= \kappa'\alpha(\beta'\alpha)^{-1} - \kappa' Ca(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1} = 0.
\end{aligned}
$$

Proof of (27): We find from (26) that

$$
C_\dagger = I_p - (\alpha A_{11}\beta' + aA_{21}\beta' + \alpha A_{12}\kappa' + aA_{22}\kappa').
$$

where, using the notation

$$
P = \alpha(\beta'\alpha)^{-1}\beta' = I_p - C \text{ and } Q = a(\kappa' Ca)^{-1}\kappa' \tag{29}
$$

we find

$$
\begin{aligned}
\alpha A_{11}\beta' &= \alpha(\beta'\alpha)^{-1}\beta' + \alpha(\beta'\alpha)^{-1}\beta' a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1}\beta' = P + PQP,\\
aA_{21}\beta' &= -a(\kappa' Ca)^{-1}\kappa'\alpha(\beta'\alpha)^{-1}\beta' = -QP\\
\alpha A_{12}\kappa' &= -\alpha(\beta'\alpha)^{-1}\beta' a(\kappa' Ca)^{-1}\kappa' = -PQ\\
aA_{22}\kappa &= a(\kappa' Ca)^{-1}\kappa = Q.
\end{aligned}
$$

It follows that

$$
C_\dagger = I_p - (P + PQP) + QP + PQ - Q) = C + (I_p - C)QC - QC = C - CQC = C - Ca(\kappa' Ca)^{-1}\kappa' C,
$$

which proves (27).

Proof of (28): We find, using P and Q from (29) that

$$\alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger = (\alpha, a)\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} \mu \\ \kappa^* \end{pmatrix} = (\alpha A_{11} + aA_{21})\mu + (\alpha A_{12} + aA_{22})\kappa^*$$
$$= (I_p - CQ)\alpha(\beta'\alpha)^{-1}\mu + Ca(\kappa' Ca)^{-1}\kappa^*,$$

because

$$\alpha A_{11} + aA_{21} = (I_p + PQ - Q)\alpha(\beta'\alpha)^{-1} = (I_p - CQ)\alpha(\beta'\alpha)^{-1},$$
$$\alpha A_{12} + aA_{22} = -Pa(\kappa' Ca)^{-1} + a(\kappa' Ca)^{-1} = Ca(\kappa' Ca)^{-1}.$$

Then

$$\alpha_\dagger(\beta_\dagger'\alpha_\dagger)^{-1}\mu_\dagger = (I_p - CQ)\alpha(\beta'\alpha)^{-1}\mu + Ca(\kappa' Ca)^{-1}\kappa^*$$
$$= (I_p - a(\kappa' Ca)^{-1}\kappa')\alpha(\beta'\alpha)^{-1}\mu + Ca(\kappa' Ca)^{-1}\kappa^*,$$

which proves (28). ■

8.2 The companion form of the stacked process $\tilde{x}_t$

We assume that x_t is p dimensional, for $t = 1, \ldots, T$, and given by the CVAR(k)

$$\Delta x_{t+1} - \gamma = \alpha(\beta'(x_t - \gamma t) - \mu) + \sum_{i=1}^{k-1}\Gamma_i(\Delta x_{t+1-i} - \gamma) + \varepsilon_{t+1}, \tag{30}$$

where α and β are $p \times r$ matrices, $\Gamma_1, \ldots, \Gamma_{k-1}$ are $p \times p$ matrices, μ is a p vector and γ an r vector.

We define the matrix polynomial

$$A(z) = I_p(1 - z) - \alpha\beta' z - \sum_{i=1}^{k-1}\Gamma_i(1 - z)z^i, \tag{31}$$

and $\Gamma = I_p - \sum_{i=1}^{k-1}\Gamma_i = A(0)$, and assume that the roots of $\det(A(z)) = 0$ satisfy $|z| > 1$ or $z = 1$. Then Δx_t and $\beta' x_t$ are stationary, if and only if $\alpha_\perp'\Gamma\beta_\perp$ has full rank $p - r$, see Johansen 1996, Theorem 4.2.

We define the stacked process

$$\tilde{x}_t' = (x_t' - \gamma' t, \ldots, x_{t-k+1}' - \gamma'(t - k + 1)),\ \tilde{\varepsilon}_t' = (\varepsilon_t', 0, \ldots, 0),$$

and the corresponding parameters

$$\tilde{\alpha}_\perp' = \alpha_\perp'(I_p, -\Gamma_1, \ldots, \Gamma_{k-1}), \tilde{\beta}_\perp' = \beta_\perp'(I_p, I_p, \ldots, I_p),\ \tilde{\mu}' = (\mu', 0, \ldots, 0). \tag{32}$$

Then the $pk \times (r + (k-1)p)$ matrices $\tilde{\alpha}$ and $\tilde{\beta}$ are

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \Gamma_1 & \dots & \Gamma_{k-1} \\ 0 & I_p & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & I_p \end{pmatrix} \tilde{\beta} = \begin{pmatrix} \beta & I_p & \dots & 0 \\ 0 & -I_p & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -I_p \end{pmatrix}. \tag{33}$$

We define the unit vectors a, b, c of dimension p, which pick out the instrument, $a'x_t$, the target, $b'x_t$, and the intermediate target $c'x_t$. We use the notation for the extended instrument, target and control vectors of dimension pk :

$$\tilde{a}' = (a', 0, \dots, 0),\ \tilde{b}' = (b', 0, \dots, 0),\ \tilde{\kappa}' = (\kappa', 0, \dots, 0). \tag{34}$$

The stacked process $\tilde{x}_{t+1}$ satisfies the equation

$$\tilde{x}_{t+1} = (I_{pk} + \tilde{\alpha}\tilde{\beta}')\tilde{x}_t - \tilde{\alpha}\tilde{\mu} + \tilde{\varepsilon}_{t+1}, \tag{35}$$

and the long-run $pk \times pk$ matrix for the stacked process is

$$\tilde{C} = (I_p, I_p, \dots, I_p)'\beta_\perp(\alpha_\perp'\Gamma\beta_\perp)^{-1}\alpha_\perp'(I_p, -\Gamma_1, \dots, -\Gamma_{k-1}). \tag{36}$$

8.3 Controlling the stacked process $\tilde{x}_t$

The control rule for the model with one lag is defined by (7), and for the stacked process we use the control rule

$$\tilde{a}(\tilde{\kappa}'\tilde{x}_t - \kappa^*), \tag{37}$$

see (34), such that x_t is changed to $x_t + a(\tilde{\kappa}'x_t - \kappa^*)$, and the lagged values $x_{t-1}, \dots, x_{t-k+1}$ are not changed, only used in $\tilde{\kappa}'x_t$.

We use this control rule to define the new process, see (8), using the equations

$$\tilde{x}_{t+1}^{new} = (I_{kp} + \tilde{\alpha}\tilde{\beta}')(\tilde{x}_t^{new} + \tilde{a}(\tilde{\kappa}'\tilde{x}_t^{new} - \tilde{\kappa}^*)) - \tilde{\alpha}\tilde{\mu} + \tilde{\varepsilon}_{t+1}.$$

Theorem 3 *The process $\tilde{x}_t^{new}$ satisfies the equations of a VAR(1):*

$$\Delta\tilde{x}_{t+1}^{new} = \tilde{\alpha}_\dagger(\tilde{\beta}_\dagger'\tilde{x}_t^{new} - \tilde{\mu}_\dagger) + \tilde{\varepsilon}_{t+1}, \tag{38}$$

for the $kp \times (r + 1 + (k-1)p)$ matrices $\tilde{\alpha}_\dagger$, $\tilde{\beta}_\dagger$, and the $pk + 1$ vector $\tilde{\mu}_\dagger$ given by

$$\tilde{\alpha}_\dagger = (\tilde{\alpha}, \tilde{a}) \begin{pmatrix} I_{r+(k-1)p} & \tilde{\beta}'\tilde{a} \\ 0_{1\times(r+(k-1)p)} & 1 \end{pmatrix},\ \tilde{\beta}_\dagger = (\tilde{\beta}, \tilde{\kappa}),\ \tilde{\mu}_\dagger = \begin{pmatrix} \tilde{\mu} \\ \kappa^* \end{pmatrix}. \tag{39}$$

The stacked process $\tilde{x}_{t+1}^{new}$ is an $I(1)$ process with $r + 1 + (k-1)p$ cointegrating relations if and only if

$$|1 + eig \begin{pmatrix} \tilde{\beta}'\tilde{\alpha} & (I_{r+(k-1)p} + \tilde{\beta}'\tilde{\alpha})\tilde{\beta}'\tilde{a} \\ \tilde{\kappa}'\tilde{\alpha} & \tilde{\kappa}'(I_{kp} + \tilde{\alpha}\tilde{\beta}')\tilde{a} \end{pmatrix}| < 1. \tag{40}$$

If this holds, the long-run value of $\tilde{x}_t^{new}$ *is, see (5),*

$$L_\infty(\tilde{x}_t^{new}) = \tilde{C}_\dagger \tilde{x}_t^{new} + \tilde{\alpha}_\dagger(\tilde{\beta}_\dagger'\tilde{\alpha}_\dagger)^{-1}\tilde{\mu}_\dagger, \tag{41}$$

for

$$\tilde{C}_\dagger = (I_p, , \ldots, I_p)'(C - Ca(\kappa' Ca)^{-1}\kappa' C)(I_p, -\Gamma_1, \ldots, -\Gamma_{k-1}), \tag{42}$$

$$\tilde{\alpha}_\dagger(\tilde{\beta}_\dagger'\tilde{\alpha}_\dagger)^{-1}\tilde{\mu}_\dagger = (I_{pk} - \tilde{C}_\dagger\tilde{a}(\tilde{\kappa}'\tilde{C}_\dagger\tilde{a})^{-1}\tilde{\kappa}')\tilde{\alpha}(\tilde{\beta}'\tilde{\alpha})^{-1}\tilde{\mu} + \tilde{C}_\dagger\tilde{a}(\tilde{\kappa}'\tilde{C}_\dagger\tilde{a})^{-1}\kappa^*. \tag{43}$$

Proof of Theorem 3. The expressions (38), (40) and (41) follow from Theorem (1) applied to the CVAR(1) process $\tilde{x}_t^{new}$. The expressions for $\tilde{C}_\dagger$ and $\tilde{\alpha}_\dagger(\tilde{\beta}_\dagger'\tilde{\alpha}_\dagger)^{-1}\tilde{\mu}_\dagger$ are given in Lemma 2, see (27) and (28), applied to the stacked process, for $\tilde{a}' = (a', 0, \ldots, 0)$ and $\tilde{\kappa}' = (\kappa_1', \kappa_2', \ldots, \kappa_{k-1}')$. ■

To illustrate the results in Theorem 3, we show in the next Corollary, how we can choose a control rule, using the idea of Figure 1 for a CVAR(2) model, as done in Corollary 1 for the case of one lag. For a CVAR(2), we find the matrices for the stacked process, $\tilde{x}_t$,

$$\tilde{\alpha} = \begin{pmatrix} \alpha & \Gamma_1 \\ 0 & I_p \end{pmatrix}, \tilde{\beta} = \begin{pmatrix} \beta & I_p \\ 0 & -I_p \end{pmatrix}, \tilde{C} = \begin{pmatrix} C & -C\Gamma_1 \\ C & -C\Gamma_1 \end{pmatrix}$$

and for the new process, $\tilde{x}_t^{new}$,

$$\tilde{\alpha}_\dagger = (\tilde{\alpha}, \tilde{a})\begin{pmatrix} I_{r+p} & \tilde{\beta}'\tilde{a} \\ 0_{1\times(r+k)} & 1 \end{pmatrix}, \tilde{\beta}_\dagger = (\tilde{\beta}, \tilde{\kappa}), \tilde{\mu}_\dagger = \begin{pmatrix} \tilde{\mu} \\ \kappa^* \end{pmatrix}$$

$$\tilde{C}_\dagger = (I_p, I_p)'(C - Ca(\kappa' Ca)^{-1}\kappa' C)(I_p, -\Gamma_1).$$

Corollary 3 *We consider the CVAR with two lags,*

$$\Delta x_t = \alpha(\beta' x_{t-1} - \mu) + \Gamma_1 x_{t-1} + \varepsilon_t,$$

see (30) and find that the long-run value of x_{t+h}*, using the control rule* $\kappa_1' x_t + \kappa_2' x_{t-1} - \kappa^*$ *at a given time* t*, is*

$$L_\infty(x_t + \kappa_1' x_t + \kappa_2' x_{t-1} - \kappa^*) = C(x_t + a(\kappa_1' x_t + \kappa_2' x_{t-1} - \kappa^*)) - C\Gamma_1 x_{t-1} + \mu^*,$$

where the constant term is

$$\mu^* = (\alpha, \Gamma_1)(\tilde{\beta}'\tilde{\alpha})^{-1}\begin{pmatrix} \mu \\ 0 \end{pmatrix}.$$

We choose the control rule $(\kappa_1, \kappa_2, \kappa^*)$ *such that the long-run value satisfies*

$$b' C(x_t + a(\kappa_1' x_t + \kappa_2' x_{t-1} - \kappa^*)) - b' C\Gamma_1 x_{t-1} + b'\mu^* = b^*,$$

that is

$$\kappa_1' = -(b' Ca)^{-1} b' C, \ \kappa_2' = (b' Ca)^{-1} b' C\Gamma_1, \ \text{and } \kappa^* = (b' Ca)^{-1}(b^* - b'\mu^*). \tag{44}$$

Applying this control rule at all times generates x_t^{new}*, see (8), which by Theorem 3 is a CVAR(1) process with cointegrating space* $sp(\tilde{\beta}, \tilde{b})$ *and adjustment space* $sp(\tilde{\alpha}, \tilde{a})$*, and the intervention is represented as the difference between two stationary processes with mean zero, just like for the lag one model (20):*

$$\tilde{\kappa}'\tilde{x}_t^{new} - \kappa^* = (b'Ca)^{-1}(b'(\alpha, \Gamma_1) \begin{pmatrix} \beta'\alpha & \beta'\Gamma_1 \\ \alpha & \Gamma_1 - I_p \end{pmatrix}^{-1} \begin{pmatrix} \beta' x_t^{new} - \mu \\ \Delta x_t^{new} \end{pmatrix} - (b' x_t^{new} - b^*)).$$

Proof. To apply Theorem 3 we need to show that condition (40) is satisfied by showing that $\tilde{\kappa}'\tilde{\alpha} = 0$ and $1 + \tilde{\kappa}'(I_{kp} + \tilde{\alpha}\tilde{\beta}')\tilde{a} = 1 + \tilde{\kappa}'\tilde{a} = 0$. This follows from

$$\tilde{\kappa}'\tilde{\alpha} = (b'Ca)^{-1} b'(-C, C\Gamma_1) \begin{pmatrix} \alpha & \Gamma_1 \\ 0 & I_p \end{pmatrix} = 0,$$

$$\tilde{\kappa}'\tilde{a} = \kappa_1' a = -(b'Ca)^{-1} b'Ca = -1.$$

■

8.4 Acknowledments

We would like to thank Takamitsu Kurita for his interst in our paper and for providing useful comments and some references to previous work.

9 References

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